Digital Signal Processing
BEC505
Chapter 1: Introduction

What is a Signal?

Anything which carries information is a signal. e.g. human voice, chirping of birds, smoke signals, gestures (sign language), fragrances of the flowers.

Many of our body functions are regulated by chemical signals, blind people use sense of touch. Bees communicate by their dancing pattern.

Modern high speed signals are: voltage changer in a telephone wire, the electromagnetic field emanating from a transmitting antenna,variation of light intensity in an optical fiber.

Thus we see that there is an almost endless variety of signals and a large number of ways in which signals are carried from on place to another place.

Signals: The Mathematical Way

A signal is a real (or complex) valued function of one or more real variable(s).When the function depends on a single variable, the signal is said to be one-dimensional and when the function depends on two or more variables, the signal is said to be multidimensional.

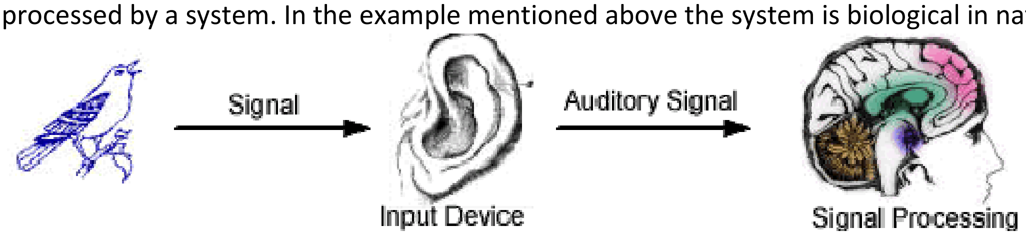
Examples of a one dimensional signal: A speech signal, daily maximum temperature, annual rainfall at a place

An example of a two dimensional signal: An image is a two dimensional signal, vertical and horizontal coordinates representing the two dimensions.

Four Dimensions: Our physical world is four dimensional(three spatial and one temporal).

What is Signal processing?

Processing means operating in some fashion on a signal to extract some useful information e.g. we use our ears as input device and then auditory pathways in the brain to extract the information. The signal is processed by a system. In the example mentioned above the system is biological in nature.



The signal processor may be an electronic system, a mechanical system or even it might be a computer program.

Analog versus digital signal processing

The signal processing operations involved in many applications like communication systems, control systems, instrumentation, biomedical signal processing etc can be implemented in two different ways

Analog or continuous time method

Digital or discrete time method..

Analog signal processing

Uses analog circuit elements such as resistors, capacitors, transistors, diodes etc

Based on natural ability of the analog system to solve differential equations that describe a physical system

The solutions are obtained in real time...

Digital signal processing

The word digital in digital signal processing means that the processing is done either by a digital hardware or by a digital computer.

Relies on numerical calculations

The method may or may not give results in real time..

The advantages of digital approach over analog approach
Flexibility: Same hardware can be used to do various kind of signal processing operation,while in the case of analog signal processing one has to design a system for each kind of operation
Repeatability: The same signal processing operation can be repeated again and again giving same results, while in analog systems there may be parameter variation due to change in temperature or supply voltage.
The choice of choosing between analog or digital signal processing depends on the application. One has to compare design time,size and thecost of the implementation.

Classification of signals

We use the term signal to mean a real or complex valued function of real variable(s) and denote the signal by x(t)
The variable t is called independent variable and the value x of t as dependent variable.
When t takes a vales in a countable set the signal is called a discrete time signal. For example
t ε {0, T, 2T, 3T, 4T,...}
t ε {....-1, 0 ,1,...}
t ε {1/2, 3/2, 5/2, 7/2,...}
For convenience of presentation we use the notation x[n] to denote discrete time signal. When both the dependent and independent variables take values in countable sets (two sets can be quite different) the signal is called Digital Signal.
When both the dependent and independent variable take value in continous set interval, the signal is called an Analog Signal.

Notation:

When we write x(t) it has two meanings. One is value of x at time t and the other is the pairs (x(t), t) allowable value of t. By signal we mean the second interpretation.

Notation for continous time signal

{x(t)} denotes the continuous time signal. Here {x(t)} is short notation for {x(t), t ε I } where I is the set in which t takes the value.

Notation for discrete time signal

Similarly for discrete time signal we will use the notation {x(t)}, where {x(t)} is short for {x(t), n ε I }.
Note that in {x(t)} and {x[n]} are dummy variables ie. {x[n]} and {x[t]} refer to the same signal. Some books use the notation x [.] to denote {x[n]} and x[n] to denote value of x at time n.
{x(t)} refers to the whole waveform,while x[n] refers to a particular value.
Most of the books do not make this distinction clean and use x[n] to denote signal and x[n0] to denote a particular value.

Discrete Time Signal Processing and Digital Signal Processing

When we use digital computers to do processing we are doing digital signal processing. But most of the theory is for discrete time signal processing where dependent variable generally is continuous. This is because of the mathematical simplicity of discrete time signal processing. Digital Signal Processing tries to implement this as closely as possible. Thus what we study is mostly discrete time signal processing and what is really implemented is digital signal processing.

Elementary Signals

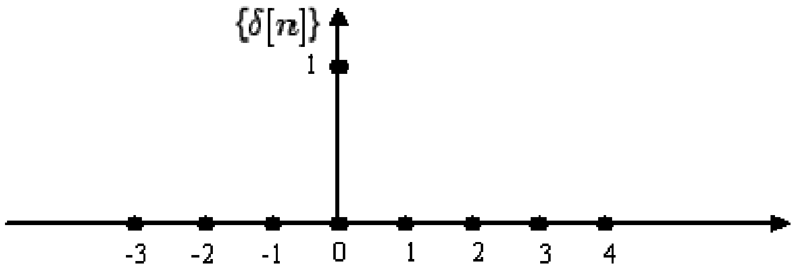
There are several elementary signals that occur prominently in the study of digital signals and digital signal processing.

(a) UNIT SAMPLE SEQUENCE: $\delta[n]$

Defined by

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Graphically this is as shown below.



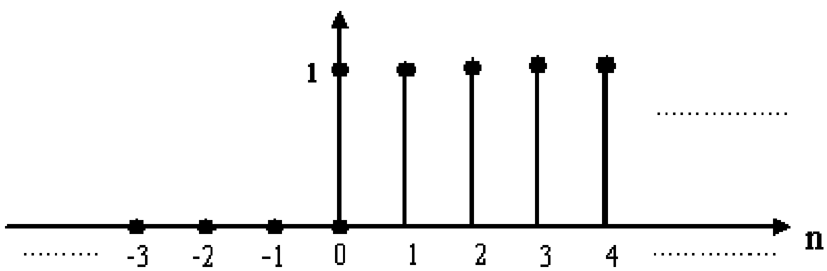
Unit sample sequence is also known as **impulse sequence.**
This plays role akin to the impulse function $\{\delta(t)\}$ of continous time. The continues time impulse $\{\delta(t)\}$ is purely a mathematical construct while in discrete time we can actually generate the impulse sequence.

(b) UNIT STEP SEQUENCE: $\{u[n]\}$

Defined by :

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Graphically this is as shown below



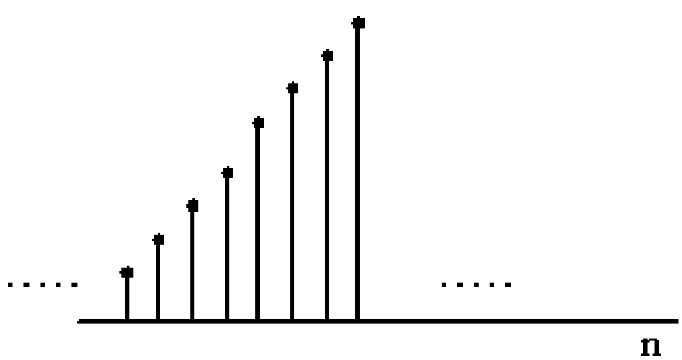
(c) EXPONENTIALSEQUENCE:

The complex exponential signal or sequence {x[n]} is defined by $x[n] = C\,\alpha^n$
where C and α are, in general, complex numbers.

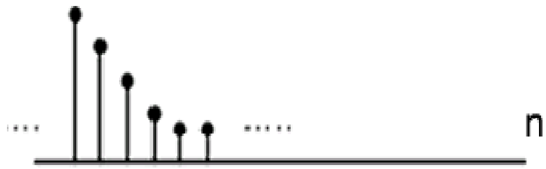
Note that by writing $\alpha = e^{\beta}$, we can write the exponential sequence as $x[n] = c\,e^{\beta n}$

Real exponential signals:

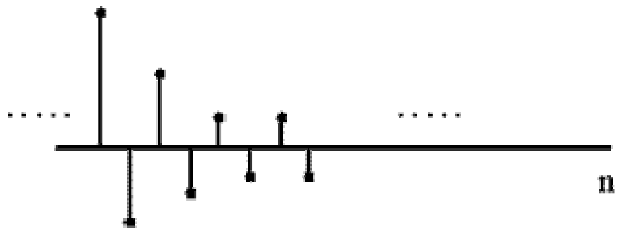
: If C and α are real, we can have one of the several type of behavior illustrated below



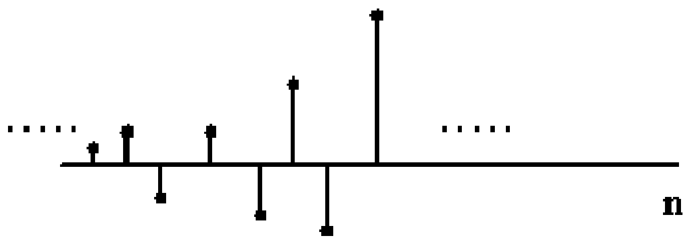
$\{x[n] = \alpha^n, \alpha > 1\}$



$\{x[n] = \alpha^n, 0 < \alpha < 1\}$



$\{x[n] = \alpha^n, -1 < \alpha < 0\}$



$\{x[n] = \alpha^n, \alpha < -1\}$

For $|\alpha| > 1$ magnitude of the signals grows exponentially, $|\alpha| < 1$ It is decaying exponential.
For $\alpha > 1$ all terms of $\{x[n]\}$ have same sign, $\alpha < 1$ sign of terms in $\{x[n]\}$ alternates.

(d)SINUSOIDAL SIGNAL:

The sinusoidal signal $\{x[n]\}$ is defined by

$$x[n] = A\cos(w_0 n + \phi)$$

Euler's relation allows us to relate complex exponentials and sinusoids as

$$e^{j w_0 n} = \cos w_0 n + j \sin w_0 n$$

and

$$A\cos(w_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j w_0 n} + \frac{A}{2} e^{-j\phi} e^{-j w_0 n}$$

The general discrete time complex exponential can be written in terms of real exponential and sinusiodal signals.

Specifically if we write C and α in polar form $C = |C| e^{j\theta}$ and $\alpha = |\alpha| e^{j w_0}$ then

$$C\alpha^n = |C||\alpha|^n \cos(w_0 n + \theta) + j|C||\alpha|^n \sin(w_0 n + \theta)$$

Thus for $|\alpha| = 1$, the real and imaginary parts of a complex exponential sequence are sinusoidal.
$|\alpha| < 1$, they correspond to sinusoidal sequence multiplied by a decaying exponential,
$|\alpha| > 1$, they correspond to sinusiodal sequence multiplied by a growing exponential.

Generating Signals with MATLAB

MATLAB, acronym for MATrix LABoratory has become a very popular software environment for complex based study of signals and systems. Here we give some sample programmes to generate the elementary signals discussed above. For details one should consider MATLAB manual or read help files.

In MATLAB, ones(M,N) is an M-by-N matrix of ones, and zeros(M,N) is an M-by-N matrix of zeros. We may use those two matrices to generate impulse and step sequence.

The following is a program to generate and display impulse sequence.

```
>> %Program to generate and display impulse response sequence
>> n = -49 : 49;
>> delta = [zeros(1,49),1,zeros(1,49)];
>> stem(n,delta) ;
```

Here **>>** indicates the MATLAB prompt to type in a command, **stem(n,x)** depicts the data contained in vector **x**as a discrete time signal at time values defined by n. One can add title and lable the axes by suitable commands. To generate step sequence we can use the following program

```
>> %Program to generate and display unit step function
>> n = -49:49;
>> u = [zeros(1,49),ones(1,50)];
>> stem(n,u);
```

We can use the following program to generate real exponential sequence

```
>> % Program to generate real exponential sequence
>> C = 1;
>> alpha = 0.8;
>> n = -10:10;
>> x = C * alpha .^ n
>> stem(n,x);
```

Note that, in this program, the base alpha is a scalar but the exponent is a vector, hence use of the operator **.∧** to denote element-by-element power.

Recap	
In last lecture you have learnt the following	
	Signals are functions of one or more independent variables.
	Systems are physical models which gives out an output signal in response to an input signals.
	Trying to identify real-life examples as models of signals and systems, would help us in understanding the subject better.

Objectives

In this lecture you will learn the following

In this chapter we will learn some of the operations performed on the sequences.

- Sequence
  Addition
  Scalar Multiplication
  Sequence Multiplication
  Shifting
  Reflection

we will learn some of the properties of signals.

- Energy of a signal
  Power of a signal
  Periodicity of signals
  Even and Odd signals
  Periodicity property of sinusoidal signals

Sequence addition: Let {x[n]} and {y[n]}be two sequences. The sequence addition is defined as term by term addition. Let {z[n]} be the resulting sequence

$$\{z[n]\} = \{x[n]\} + \{y[n]\}$$

where each term $z[n] = x[n] + y[n]$We will use the following notation

$$\{x[n]\} + \{y[n]\} = \{x[n] + y[n]\}$$

Scalar multiplication:Let *a* be a scalar. We will take *a* to be real if we consider only the real valued signals, and take a to be a complex number if we are considering complex valued sequence. Unless otherwise stated we will consider complex valued sequences. Let the resulting sequence be denoted by {w[n]}

$$\{w[n]\} = a\{x[n]\}$$

is defined by

$$w[n] = ax[n]$$

each term is multiplied by *a* We will use the notation

$$a\{w[n]\} = \{aw[n]\}$$

Note: If we take the set of all sequences and define these two operations as addition and scalar multiplication they satisfy all the properties of a linear vector space.

Sequence multiplication:

Let {x[n]} and {y[n]} be two sequences, and {z[n]} be resulting sequence

$$\{z[n]\} = \{x[n]\}\{y[n]\}$$

where $z[n] = x[n]\,y[n]$

The notation used for this will be $\{x[n]\}\{y[n]\} = \{x[n]\,y[n]\}$

Now we consider some operations based on independent variable n.

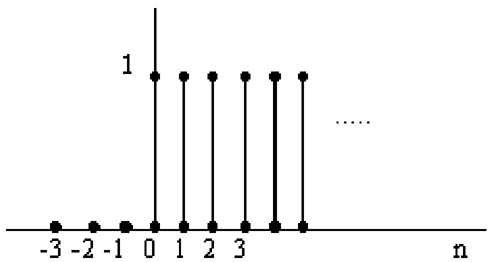
Shifting:

This is also known as translation. Let us shift a sequence {x[n]} by n_0 units, and the resulting sequence be{y[n]}

$$\{y[n]\} = z^{-n_0}(\{x[n]\})$$

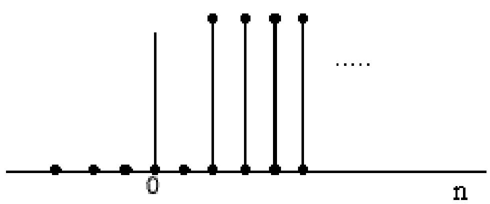
where $z^{-n_0}()$ is the operation of shifting the sequence right by n_0 unit. The terms are defined by y[n] = x[n - n_0]. We will use short notation {x[n - n_0]} to denote shift by n_0.
Figure below show some examples of shifting.

{x[n]}



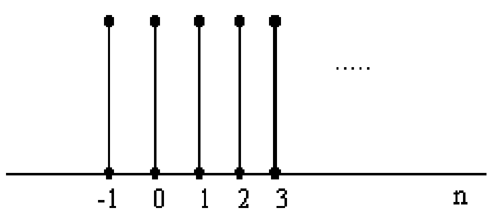
Consider the figure to the left.

{x[n-2]}



A negative value of n_0 means shift towards right.

{x [n+1]}



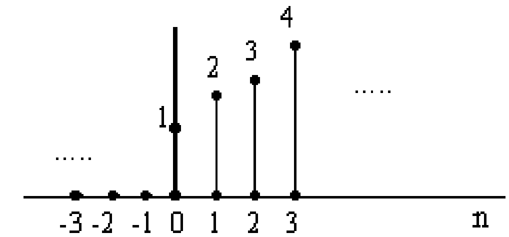
A positive value of n_0 means shift towards left.

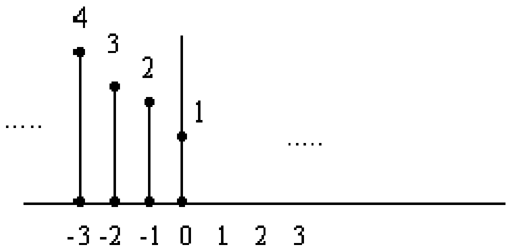
Reflection:

Let {x[n]} be the original sequence, and {y[n]} be reflected sequence, then y[n] is defined by

$$y[n] = x[-n]$$

{x[n]}



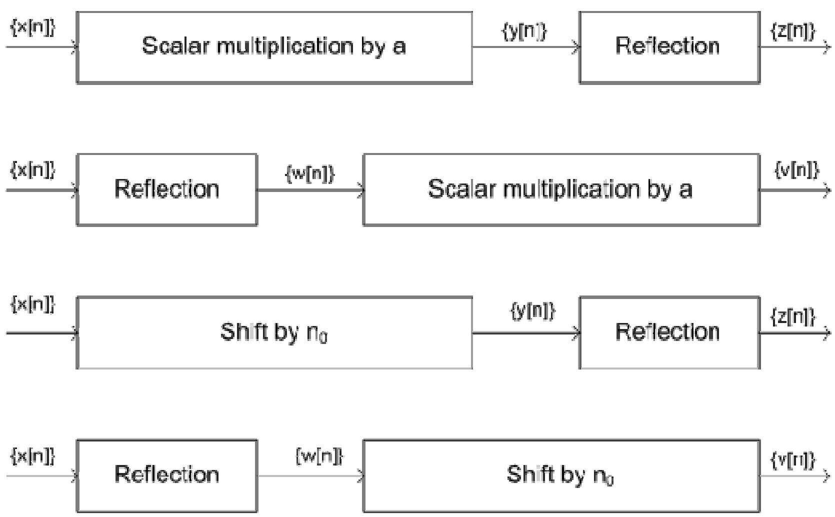


We will denote this by {x[-n]}

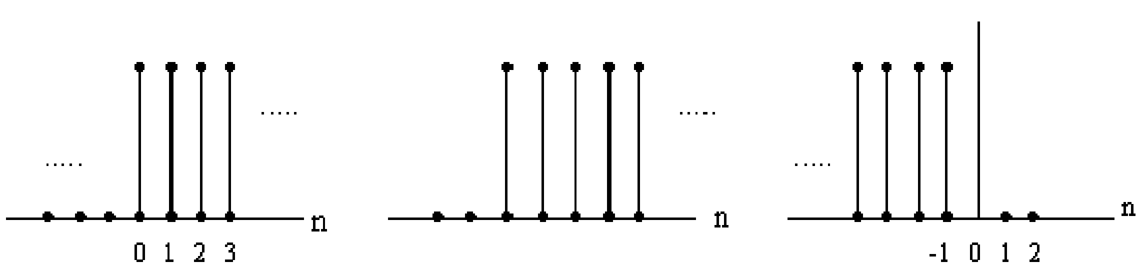
When we have complex valued signals, sometimes we reflect and do the complex conjugation, ie, y[n] is defined by y[n] = x*[-n], where * denotes complex conjugation. This sequence will be denoted by {x*[-n]}.

We will learn about more complex operations later on. Some of these operations commute, ie. if we apply two operations we can interchange their order and some do not commute. For example scalar multiplication and reflection commute.

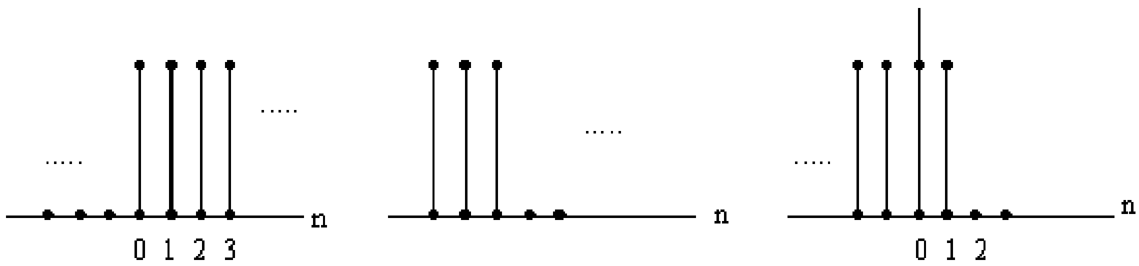
Then v[n] = z[n] for all n. Shifting and scaling do not commute.



{x[n]} {y[n]} = {x[n-1} {z[n]} = {y[-n]}



{x[n]}　　　　{w[n]} = {x[-n]}　　　　{u[n]} = {w[n-1]}



We can combine many of these operations in one step, for example {y[n]} may be defined as y[n] = 2x [3-n].

Some Properties of signals

Energy of a Signal:

The total energy of a signal {x[n]} is defined by

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

A signal is referred to as an energy signal, if and only if the total energy of the signal E_x is finite.

Power of a signal: If {x[n]} is a signal whose energy is not finite, we define power of the signal as

$$P_x = \lim_{N \to \infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2$$

A signal is referred to as a power signal if the power P_x satisfies the condition

$$0 \leqslant P_x < \infty$$

An energy signal has a zero power and a power signal has infinite energy. There are signals which are neither energy signals nor power signals. For example {x[n]} defined by x[n] = n does not have finite power or energy.

Periodic Signals:

An important class of signals that we encounter frequently is the class of periodic signals. We say that a signal {x[n] is periodic with period N, where N is a positive integer, if the signal is unchanged by the time shift of N ie.,

{x[n]} = {x[n + N]}

or x[n] = x[n + N] for all n.

Since {x[n]} is same as {x[n+N]} , it is also periodic so we get

{x[n]} = {x[n+N]} = {x[n+N+N]} = {x[n+2N]}

Generalizing this we get {x[n]} = {x[n+kN]}, where k is a positive integer. From this we see that {x[n]} is periodic with 2N, 3N,... The fundamental period N_0 is the smallest positive value N for which the signal is periodic.

The signal illustrated below is periodic with fundamental period $N_0 = 4$

By change of variable we can write {x[n]} = {x[n+N]} as {x[m-N]} = {x[m]} and then arguing as before, we see that

$$\{x[n]\} = \{x[n+kN]\},$$

for all integer values of k positive, negative or zero. By definition, period of a signal is always a positive integer N.
Except for a all zero signal all periodic signals have infinite energy. They may have finite power. Let {x[n]} be periodic with period N, then the power P_x is given by

$$P_x = \lim_{M\to\infty} \frac{1}{(2M+1)} \sum_{n=-M}^{M} |x[n]|^2$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \Big[\sum_{n=0}^{N-1} |x[n]|^2 + \sum_{n=N}^{2N-1} |x[n]|^2 + \dots + \sum_{n=(k-1)N-1}^{kN-1} |x[n]|^2$$

$$+ \sum_{n=kN}^{M} |x[n]|^2 + \sum_{n=-N}^{-1} |x[n]|^2 + \dots + \sum_{n=-kN}^{-(k-1)N-1} |x[n]|^2 + \sum_{n=-M}^{-kN-1} |x[n]|^2 \Big]$$

where k is largest integer such that $kN - 1 \le M$. Since the signal is periodic, sum over one period will be same for all terms. We see that k is approximately equal to M/N (it is integer part of this) and for large M we get 2M/N terms and limit 2M/(2M +1) as M goes to infinite is one we get

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

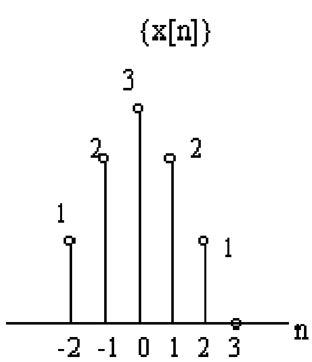
Even and odd signals:

A real valued signal {x[n]} is referred to as an even signal if it is identical to its time reversed counterpart ie, if
{x[n]} = {x[-n]}

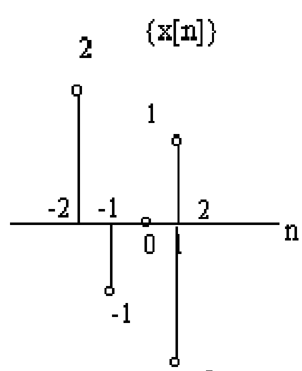
A real signal is referred to as an odd signal if

$$\{x[n]\} = \{-x[-n]\}$$

An odd signal has value 0 at n = 0 as x[0] = -x[n] = - x[0]



An even signal



An odd Signal

Given any real valued signal {x[n]} we can write it as a sum of an even signal and an odd signal.

Consider the signals

$$Ev(\{x[n]\}) = \{x_e[n]\} = \{1/2\,(x[n] + x[-n])\}$$

and $$Od(\{x[n]\}) = \{x_o[n]\} = \{1/2(x[n] - x[-n])\}$$

We can see easily that

$$\{x[n]\} = \{x_e[n]\} + \{x_o[n]\}$$

The signal $\{x_e[n]\}$ is called the even part of $\{x[n]\}$. We can verify very easily that $\{x_e[n]\}$ is an even signal. Similarly, $\{x_o[n]\}$ is called the odd part of $\{x[n]\}$ and is an odd signal. When we have complex valued signals we use a slightly different terminology. A complex valued signal $\{x[n]\}$ is referred to as a conjugate symmetric signal if

$$\{x[n]\} = \{x^*[-n]\}$$

where x^* refers to the complex conjugate of x. Here we do reflection and complex conjugation. If $\{x[n]\}$ is real valued this is same as an even signal.
A complex signal $\{x[n]\}$ is referred to as a conjugate antisymmetric signal if

$$\{x[n]\} = \{-x^*[-n]\}$$

We can express any complex valued signal as sum conjugate symmetric and conjugate antisymmetric signals. We use notation similar to above

$$Ev(\{x[n]\}) = \{x_e[n]\} = \{1/2(x[n] + x^*[-n])\}$$

and $$Od(\{[n]\}) = \{x_o[n]\} = \{1/2(x[n] - x^*[-n])\}$$

then $$\{x[n]\} = \{x_e[n]\} + \{x_o[n]\}$$

We can see easily that $\{x_e[n]\}$ is conjugate symmetric signal and $\{x_o[n]\}$ is conjugate antisymmetric signal. These definitions reduce to even and odd signals in case signals takes only real values.

Periodicity properties of sinusoidal signals:

Let us consider the signal. We see that if we replace w_0 by $(w_0 + 2\pi)$ we get the same signal. In fact the signal with frequency $w_0 \pm 2\pi$, $w_0 \pm 4\pi$ and so on. This situation is quite different from continuous time signal $\{\cos w_0 t, -\infty < t < \infty\}$ where each frequency is different. Thus in discrete time we need to consider frequency interval of length 2π only. As we increase w_0 to π signal oscillates more and more rapidly. But if we further increase frequency from π to 2π the rate of oscillations decreases. This can be seen easily by plotting signal $\{\cos w_0 n\}$ for several values of.
The signal $\{\cos w_0 n\}$ is not periodic for every value of. For the signal to be periodic with period N > 0, we should have

$$\{\cos w_0 n\} = \{cos w_0(n + N)\}$$

that is $w_0 N$ should be some multiple of 2π.

$$w_0 N = 2\pi m$$

$$\frac{w_0}{2\pi} = \frac{m}{N}$$

or

Thus signal $\{\cos w_0 n\}$ is periodic if and only if $\frac{w_0}{2\pi}$ is a rational number.
Above observations also hold for complex exponential signal $\{x[n]\} = \{e^{jw_0 n}\}$

Discrete-Time Systems:

A discrete-time system can be thought of as a transformation or operator that maps an input sequence $\{x[n]\}$ to an output sequence $\{y_k[n]\}$

$$\{x[n]\} \longrightarrow \boxed{T(.)} \longrightarrow \{y[n]\}$$

By placing various conditions on T(.) we can define different classes of systems.

Basic System Properties

- Systems with or without memory:
- Invertibility
- Causality
- Stability
- Time invariance
- Linearity

Systems with or without memory:

A system is said to be memoryless if the output for each value of the independent variable at a given time n depends only on the input value at time n. For example system specified by the relationship

$$y[n] = \cos(x[n]) + 3$$

is memoryless. A particularly simple memoryless system is the identity system defined by

$$y[n] = x[n]$$

In general we can write input-output relationship for memoryless system as

$$y[n] = g(x[n])$$

Not all systems are memoryless. A simple example of system with memory is a delay defined by

$$y[n] = x[n-1]$$

A system with memory retains or stores information about input values at times other than the current input value.

Invertibility:

A system is said to be invertible if the input signal $\{x[n]\}$ can be recovered from the output signal $\{y_k[n]\}$. For this to be true two different input signals should produce two different outputs. If some different input signal produce same output signal then by processing output we can not say which input produced the output.

Example of an invertible system is

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

then

$$x[n] = y[n] - y[n-1]$$

Example if a non-invertible system is

$$y[n] = 0$$

That is the system produces an all zero sequence for any input sequence. Since every input sequence gives all zero sequence, we can not find out which input produced the output.

The system which produces the sequence $\{x[n]\}$ from sequence $\{y_k[n]\}$ is called the inverse system. In communication system, decoder is an inverse of the encoder.

Causality :

A system is causal if the output at anytime depends only on values of the input at the present time and in the past.

$$y[n] = f(x[n], x[n-1],...)$$

All memoryless systems are causal. An accumulator system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[h]$$

is also causal. The system defined by

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^{N} x[n-k]$$

is noncausal.

For real time system where n actually denoted time causality is mportant. Causality is not an essential constraint in applications where n is not time, for example, image processing. If we are doing processing on recorded data, then also causality may not be required.

Stability :

There are several definitions for stability. Here we will consider bounded input bonded output (BIBO) stability. A system is said to be BIBO stable if every bounded input produces a bounded output. We say that a signal {x[n]} is bounded if

$$|x[n]| < M < \infty \quad \text{for all } n$$

The moving average system

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^{N} x[n]$$

is stable as y[n] is sum of finite numbers and so it is bounded. The accumulator system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is unstable. If we take {x[n]} = {u[n]}, the unit step then y[0] = 1, y[1] = 2, y[2] = 3, are y[n] = n +1, n ≥ 0 so y[n]grows without bound.

Time invariance :

A system is said to be time invariant if the behavior and characteristics of the system do not change with time.Thus a system is said to be time invariant if a time delay or time advance in the input signal leads to identical delay or advance in the output signal. Mathematically if

$$\{y[n]\} = T(\{x[n]\})$$

then $\{y[n-n_0]\} = T(\{x[n-n_0]\})$ for any n_0

Let us consider the accumulator system

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

If the input is now $\{x_1[n]\} = \{x[n-n_0]\}$ then the corresponding output is

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k]$$
$$= \sum_{k=-\infty}^{n} x[k]$$

The shifted output signal is given by

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

The two expression look different, but infact they are equal. Let us change the index of summation by $l = k - n_0$ in the first sum then we see that

$$y_1[n] = \sum_{l=-\infty}^{n-n_0} x[l]$$
$$= y[n-n_0]$$

Hence, $\{y[n]\} = \{y[n-n_0]\}$ and the system is time-invariant. As a second example consider the system defined by $y[n] = nx[n]$

if $\{x_1[n]\} = \{x[n-n_0]\}$

$$y_1[n] = nx_1[n] = nx[n-n_0]$$

while $y[n-n_0] = (n-n_0)x[n-n_0]$

and so the system is not time-invariant. It is time varying. We can also see this by giving a counter example. Suppose input is $\{x[n]\} = \{\delta[n]\}$ then output is all zero sequence. If the input is $\{\delta[n-1]\}$ then output is $\{\delta[n-1]\}$ which is definitely not a shifted version version of all zero sequence.

Linearity :

This is an important property of the system. We will see later that if we have system which is linear and time invariant then it has a very compact representation. A linear system possesses the important property of superposition: if an input consists of weighted sum of several signals, the nthe output is also weighted sum of the responses of the system to each of those input signals. Mathematically let $\{y_1[n]\}$ be the response of the system to the input $\{x_1[n]\}$ and let $\{y_2[n]\}$ be the response of the system to the input. Then the system is linear if:

Additivity: The response to $\{x[n]\} + \{x_2[n]\}$ is $\{y_1[n]\} + \{y_2[n]\}$

Homogeneity: The response to $a\{x_1[n]\}$ is $a\{y_1[n]\}$, where a is any real number if we are considering only real signals and a is any complex number if we are considering complex valued signals.

Continuity: Let us consider $\{x_1[n]\}, \{x_2[n]\}, \dots \{x_k[n]\} \dots$ be countably infinite number of signals such that

$$\lim_{k\to\infty} \{x_k[n]\} = \{x[n]\}$$

Let the corresponding output signals be denoted by $\{y_n[n]\}$ and $\lim_{k\to\infty} \{y_n[n]\} = \{y[n]\}$

We say that system posseses the continuity property if the response of the system to the limiting input $\{x[n]\}$ is limit of the responses.

$$T(\lim_{k\to\infty} \{x_k[n]\}) = \lim_{k\to\infty} T(\{ x_k[n]\})$$

The additivity and continuity properties can be replaced by requiring that system is additive for countably infinite number of signals i.e. response to

$\{x_1[n]\} + \{x_2[n]\} + ... + \{x_n[n]\} + ...$ is $\{y_1[n]\} + \{y_2[n]\} + ... + \{y_k[n]\} + ...$

Most of the books do not mention the continuity property. They state only finite additivity and homogeneity. But from finite additivity we can not deduce countable additivity. This distinction becomes very important in continuous time systems.

A system can be linear without being time invariant and it can be time invariant without being linear.

If a system is linear, an all zero input sequence will produce a all zero output sequence. Let $\{0\}$ denote the all zero sequence, then. If $T(\{x[n]\} = \{y[n]\})$ then by homogeneity property $T(0.\{x[n]\}) = 0.\{y[n]\}$

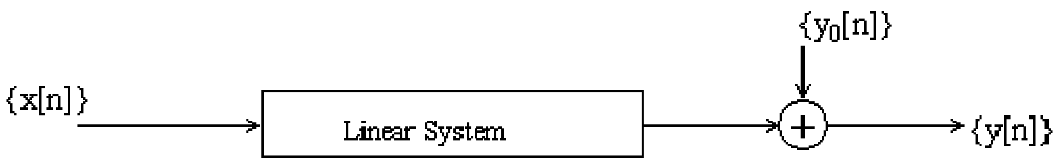
or, $T(\{0\}) = \{0\}$

Consider the system defined by $y[n] = 2x[n] + 3$

This system is not linear. This can be verified in several ways. If the input is all zero sequence $\{0\}$, the output is not an all zero sequence. Although the defining equation is a linear equation is x and y the system is nonlinear. The output of this system can be represented as sum of a linear system and another signal equal to the zero input response. In this case the linear system is

$$y[n] = 2x[n]$$

and the zero-input response is $y_0[n] = 3$ for all n



Such systems correspond to the class of incrementally linear system. System is linear in term of differnce signal i.e if we define $\{x_d[n]\} = \{x_1[n]\} - \{x_2[n]\}$ and. Then in terms of $\{x_d[n]\}$ and $\{y_d[n]\}$ the system is linear.

The Convolution Sum:

The representation of discrete time signals in terms of impulses.

The key idea is to express an arbitrary discrete time signal as weighted sum of time shifted impulses.

Consider the product of signal $\{x[n]\}$ and the impulse sequence. We know that

$$a\{\delta[n]\} = \begin{cases} a, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

and

$$b\{\delta[n-k]\} = \begin{cases} b, & n = k \\ 0, & n \neq k \end{cases}$$

Using these relations we can write

$$\{x[n]\} = \ldots x[-2]\{\delta[n+2]\} + x[-1]\{\delta[n+1]\} + x[0]\{\delta[n]\} + x[1]\{\delta[n-1]\} + x[2]\{\delta[n-2]\} + \ldots$$

$$= \sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\} \tag{4.1}$$

A graphical illustration is shown below

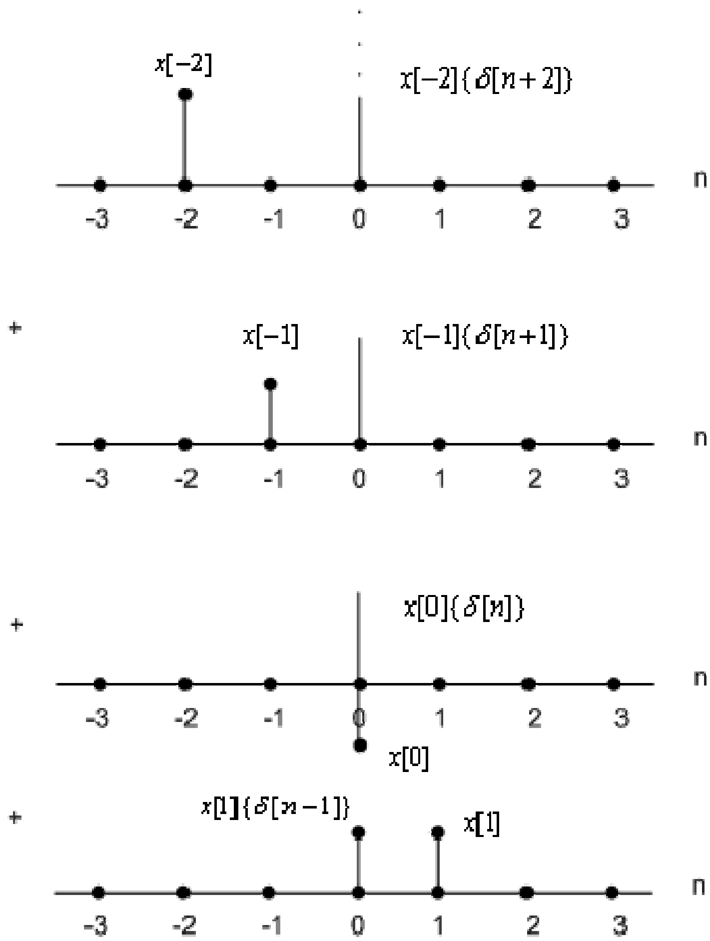


Fig 4.1

Given an arbitrary sequence we can write it as a linear combination of shifted unit impulses $\{\delta[n-k]\}$, where the weights of their combination are x[k], the k[th] term of the sequence. For any given n, in the summation

$$\sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\}$$

there is only one term which is non-zero and so we do not have to worry about the convergence.

Consider the unit step sequence {u[n]}. Since $u[n] = 0, n < 0$, and $u[n] = 1, n < 0$, it has representation

$$\{u[n]\} = \sum_{k=0}^{\infty}\{\delta[n-k]\}$$

The Discrete Time Impulse response of linear Time Invariant System:

We use linearity property of the system to represent its response in terms of its response shifted impulse sequences. The time invariance further simplifies their representation. Let $\{x[n]\}$ be the input signal and $\{y[n]\}$ be the output sequence, and T( ) represent the linear system

$$\{y[n]\} = T(\{x[n]\})$$
$$= T\left(\sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\}\right)$$

using (4.1)

Now we use the linearity property of the system we get

$$\{y[n]\} = \sum_{k=-\infty}^{\infty} x[k]T(\{\delta[n-k]\})$$
$$\{y[n]\} = T(\{x[n]\})$$

Note that without countable additivity property the last step is not justified (From finite additivity we can not get countable additivity). Let us define

$$\{h_k[n]\} = T(\{\delta[n-k]\})$$

i.e. $\{h_k[n]\}$ is the response of the system to a delayed unit sample sequence. Then we see

$$\{y[n]\} = \sum_{k=-\infty}^{\infty} x[k]\{h_k[n]\}$$

The output signal is linear combination of the signals.

In general the responses $\{h_k[n]\}$ need not be related to each other for different values of k. However, if linear system is also time-invariant, then these responses are related. Let us define impulse response (unit sample response)

$$\{h[n]\} = T(\{\delta[n]\})$$

Then

$$\{h_k[n]\} = T(\{\delta[n-k]\})$$
$$= \{h[n-k]\}$$

For the LTI system output {y[n]} is given by

$$\{y[n]\} = \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\} \qquad (4.2)$$

This result is know as convolution of sequences $\{x[n]\}$ and. Thus output signal for an LTI system is convolution of input signal $\{x[n]\}$ and the impulse response. This operation is symbolically represented by

$$\{y[n]\} = \{x[n]\} * \{h[n]\} \qquad (4.3)$$

We see that equation (4.2) expresses the response of an LTI system to an arbitrary signal in terms of the systems response to unit impulse. Thus an LTI system is completely specified by its impulse response.

The n^{th} term $y[n]$ in the equation (4.2) is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (4.4)$$

This is known as convolution sum. To convolve two sequences, we have to calculate this convolution sum for all values of n. Since right hand side is sum of infinite series, we assume that this sum is well defined.

Example:

Consider $\{x[n]\}$ and $\{h[n]\}$ shown below

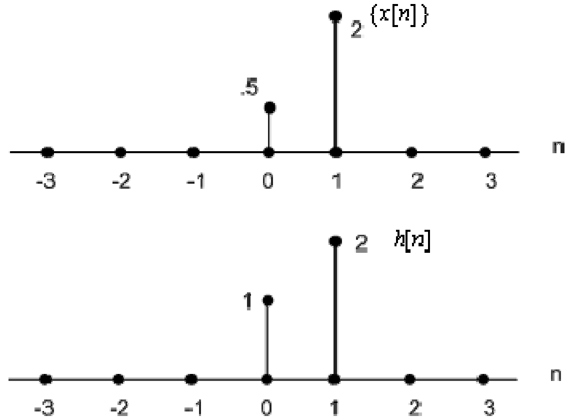


Fig 4.2

Since only $x[0]$ and $x[1]$ one non zero we have

$$\{y[n]\} = x[0]\{h[n]\} + x[1]\{h[n-1]\}$$

These one illustrated below

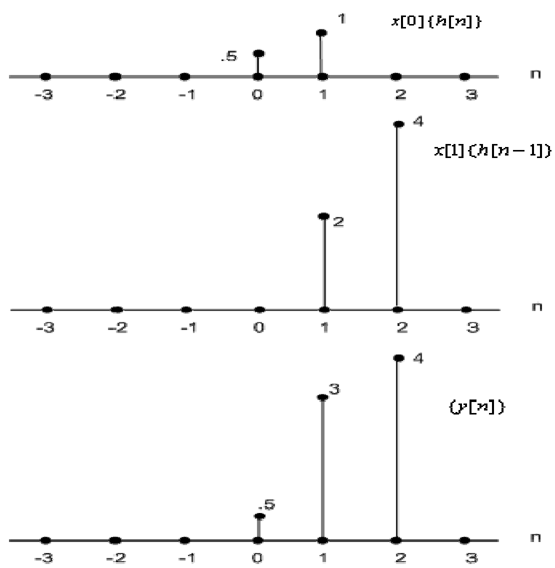


Fig 4.3

Here we have done calculation according to equation (4.2).

To do calculation according to equation (4.4) we first plot - $\{x[k]\}$ as function of k and $\{h[n-k]\}$ as function of k for some fixed values of n. Then multiply sequence $\{x[k]\}$ and $\{h[n-k]\}$ term by term to obtain sequence. Than final the sum of the terms of the sequence. This is illustrated below

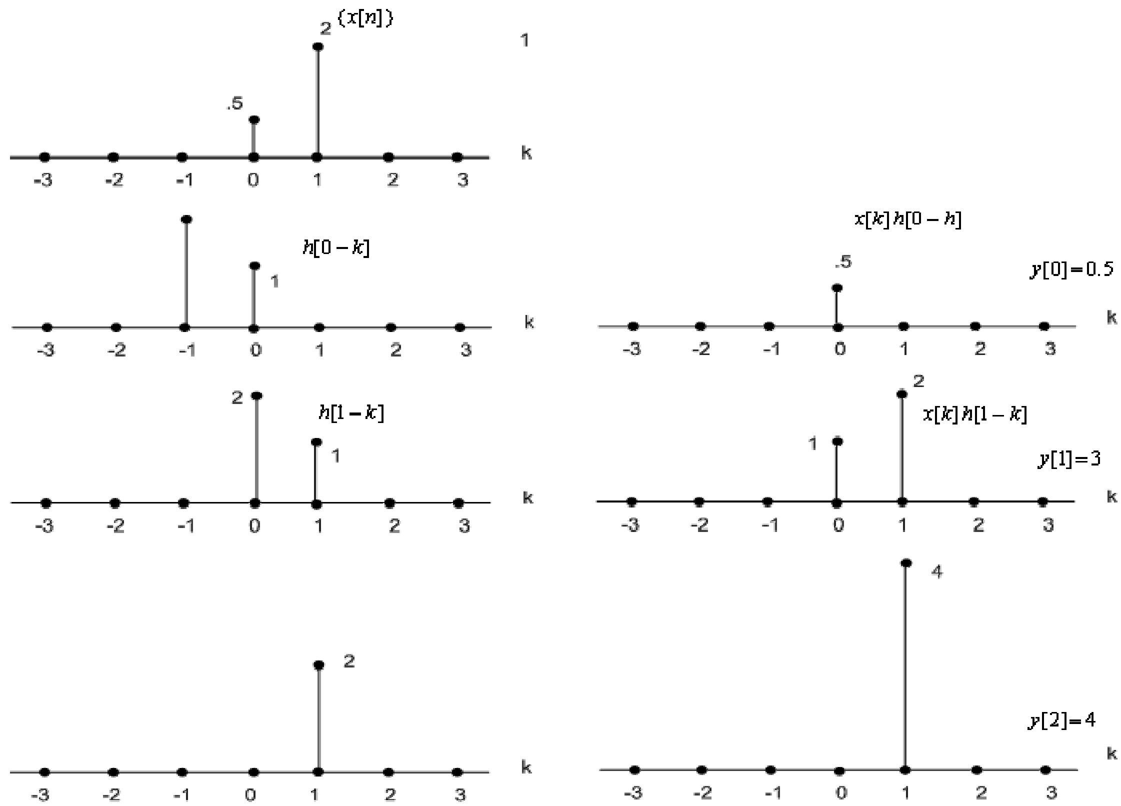


Fig 4.4

One can see easily that for other value of n $\{x[h]h[h-k]\}$ is all zero sequence and for these value of n, output is zero.

Properties of discrete-time linear convolution and system properties

If $\{x[n]\},\{y[n]\}$ and $\{z(n)\}$ are sequences, then the following useful properties of the discrete time convolution can be shown to be true

1. Commutativity

$$\{x[n]\} * \{y(n) = \{y[n]\}\} * \{x[n]\}$$

2. Associativity

$$\{x[n]\} * (\{y[n]\} * \{z[n]\}) = (\{x[n]\} * \{y[n]\}) * \{z[n]\}$$

3. Distributivity over sequence addition

$$\{x[n]\} * (\{y[n]\} + \{z[n]\}) = \{x[n]\} * \{y[n]\} + \{x[n]\} * \{z[n]\}$$

4. The identity sequence $\{\delta[n]\}$

$$\{x[n]\} * \{\delta[n]\} = \{x[n]\}$$

5. Delay operation

$$\{x[n]\} * \{\delta[n-k]\} = \{x[n-k]\}$$

6. Multiplication by a constant

$$\{ax[n]\} * \{y[n]\} = a(\{x[n] * \{y[n]\}\})$$

Note that these properties are true only if the convolution sum (4.4) exists for every n.

If the input output relation is defined by convolution i.e. if

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For a given sequence $\{h[n]\}$, then the system is linear and time invariant. This can be verified using the properties of the convolution listed above. The impulse response of the systems is obviously.
In terms of LTI system, commutative property implies that we can interchange input and impulse response.

Fig 4.5

The distributive property implies that parallel interconnection of two LTI system is an LTI system with impulse response as sum of two impulse responses.

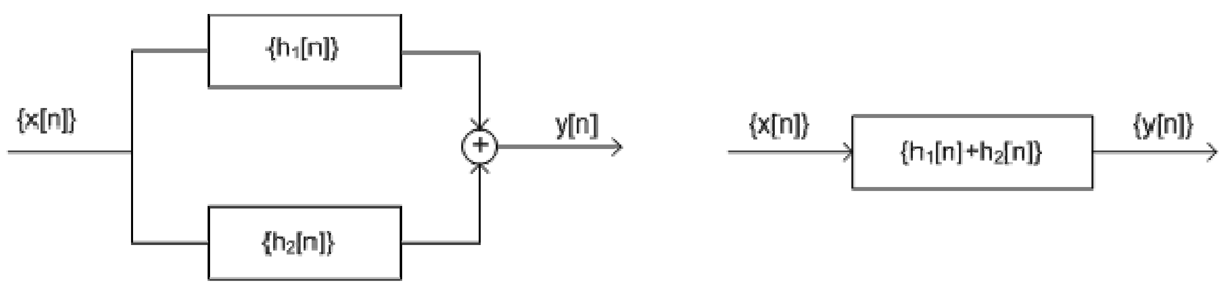


Fig 4.6

The associativity property implies that series connection of two LTI system is an LTI system. Where impulse response is convolution of individual responses. The commutativity property implies that we can interchange the order of the two system in series.

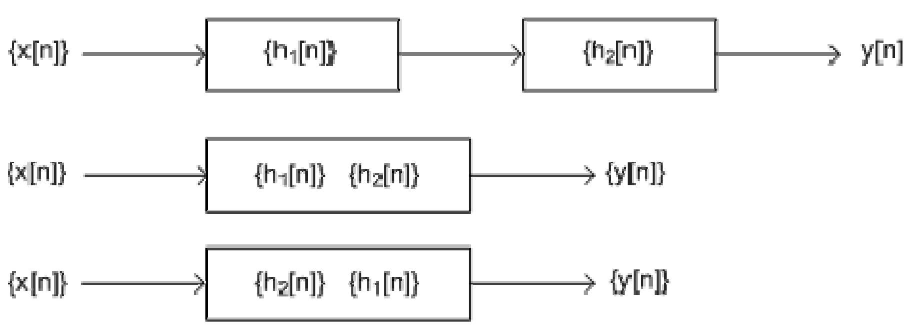


Fig 4.7

Since an LTI system is completely characterized by its impulse response, we can specify system-properties in terms of impulse response.

1. Memoryless system: From equation (4.4) we see that an LTI system is memory less if and only if.
2. Causality for LTI system: The output of a causal system depends only on preset and past-values of the input. In order for a system to be causal $y[n]$ must not depend on $x[n]$ for. From equation (4.4) we see that for this to be true, all of the terms $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero.

   $h[n-k] = 0$ for $k > n$

   put $n - k = m$ to get

   $h[m] = 0$ for $0 > n - k = m$

   or $h[m] = 0$ for $m < 0$

   Thus impulse response $\{h[n]\}$ for a causal LTI system must satisfy the condition h[n] = 0 for n < 0.

   If the impulse response satisfies this condition, the system is causal. For a causal system we can write

   $$y[n] = \sum_{k=-\infty}^{n} x[h]h[n-k]$$

   or

   $$y[n] = \sum_{k=0}^{n} h[k]x[n-k]$$

   We say a sequence $\{x[n]\}$ is causal if $x[n] = 0$, for n < 0.
3. Stability for LTI system: A system is stable if every bounded input produces a bonded output. Consider input $\{x[n]\}$ such that $|x[n]| < B$ for all n.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Taking absolute value

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right|$$

From triangle inequality for complex numbers $|z_1 + z_2| \le |z_1| + |z_2|$ we get

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$= \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

Using property that $|z_1 z_2| = |z_1||z_2|$

Since each $|x[n-k<B]|$ we get

$$|y[n]| < \sum_{k=-\infty}^{\infty} |h[k]| B$$
$$= B \sum_{k=-\infty}^{\infty} |h[k]|$$

If the impulse response is absolutely summable, that is

$$\sum_{k=-\infty}^{\infty} |h[k]| = M < \infty \tag{4.5}$$

then $|y[n]| < MB$

and $y[n]$ is bounded for all n, and hence system is stable. Therefore equation (4.5) is sufficient condition for system to be stable. This condition is also necessary. This is prove by showing that if condition (4.5) is violated then we can find a bounded input which produces an unbounded output. Let

$$\sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

Let
$$x[n] = \begin{cases} \dfrac{h^*[-n]}{|h[-n]|} & \text{if } h[-n] \neq 0 \\ 0 & \text{if } h[-n] = 0 \end{cases}$$

$$|x[n]| = \begin{cases} 1 & \text{if } h[-n] \neq 0 \\ 0 & \text{if } h[-n\} = 0 \end{cases}$$

This is a bounded sequence

$$y[0] = \sum_{k=-}^{\infty} h[k] x[-k] = \sum_{\substack{k=-\infty \\ h[k]\neq 0}}^{\infty} h[k]\, x[0k]$$
$$= \sum_{\substack{k=-\infty \\ h[k]\neq 0}}^{\infty} \frac{h[k] h^*[k]}{|h\{k\}|}$$
$$= \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

So y[0] is unbounded. Thus, the stability of a discrete time linear time invariant system is equivalent to absolute summability of the impulse response.

Causal LTI systems described by difference equations

An important subclass of linear time invariant system is one where the input and output sequences satisfy constant coefficient linear difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k$$

(4.6)

$$a_k, k = 0, 1, 2, \ldots N$$

$$b_k, k = 0, 1, 2, \ldots M$$

The constants, $\{x[n]\}$ is input sequence and $\{y[n]\}$ is output sequence. We can solve equation (4.6) in a manner analogous to the differential equation solution, but for discrete time we can use a different approach. Assume that. We can write

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k\, x[n-k] - \sum_{k=1}^{N} a_k\, y[n-k]\right\}$$

(4.7)

In order to find $y[n]$ we need previous N values of the output. Thus if we know the input sequence $\{x[n]\}$ and a set of initial condition $y[-N], y[-N+1], \ldots y[-1]$ we can find values of.

Example: Consider the difference equation

$$y[n] + 0.5y[n-1 = x[n]$$

then $y[n] = x[n] - 0.5\,y[n-1]$

Let us take $y[-1] = C$

$$y[0] = x[0] - 0.5C$$

$$y[1] = x[0] - 0.5\,y[0]$$

$$= x[1] - 0.5x[0] + (0.5)^2\,C$$

$$y[2] = x[0] - 0.5\,y[1]$$

$$= x[2] - 0.5x[1] + (0.5)^2\,x[0] + (0.5)^3\,C$$

$$y[n] = x[n] - 0.5\,x\left|n - 1 + (0.5)^2\,x[n-2] + \ldots + (-0.5)^2\,x[0] + (-0.5)^{n+1}\,C\right|$$

This system is not linear for all values of the initial condition. For a linear system all zero input sequence must produce a all zero output sequence. But if C is different from zero, then output sequence is not an all system is linear. System is not time invariant in general. Suppose input is $\{\delta[n]\}$ than we have

$$y[0] = 1 - 0.5C, y[1] = -0.5 + (0.5)^2\,C, \ldots$$

If we use input as $\{\delta[n-1]\}$ then

$$y[0] = -0.5C, y[1] = 1 + (0.5)^2\,C, \ldots$$

It is obvious that second sequence is not a shifted version of the first sequence unless. The system is linear time invariant if we assume initial rest condition, i.e. if $x[n] = 0, n < n_0$ then. With initial rest condition the system described by constant coefficient-linear difference equation is linear, time invariant and causal.

The equation of the form (4.7) is called recursive equation if $N \geq 1$, since it specifies a recursive algorithm for finding out the output sequence. In special case $N = 0$, we have

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \tag{4.8}$$

Here $y[n]$ is completely specified in terms of the input. Thus this equation is called non-recursive equation. If input $\{x[n]\} = \{\delta[n]\}$, then we see that the output is equal to impulse response

$$y[n] = h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise}. \end{cases}$$

The impulse response is non-zero for finitely many values. A system with the property that impulse response is non-zero only for finitely many values is known as finite impulse response (FIR) system. A system described by non-recursive equation is always FIR. A system described recursive equation generally has a response which is non-zero for infinite duration and such systems one known as infinite impulse response system (IIR). A system described by recessive equation may have a finite impulse response.

The Discrete Time Fourier Transform

In the previous chapter we used the time domain representation of the signal. Given any signal {x[n]} we can write it as linear combination of basic signals. Another representation of signals that has been found very useful is frequency domain representation. In the mid 1960s an algorithm for calculation of the Fourier transform was discovered, known as the Fast-Fourier Transform (FFT) algorithm. This spurred the development of digital signal processing in many areas. The Fourier representation of signals derives its importance from the fact that exponential signals are eigenfunctions for the discrete time LTI systems. What we mean by this is that if $\{z^n\}$ is input signal to an LTI system then output is given by. Let us consider an LTI system with impulse response. Then the output is given by

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} \\ &= H(z)z^n \end{aligned}$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ assuming that the summation in right-hand side converges. Thus output is same exponential sequence multiplied by a constant that depends on the value of.

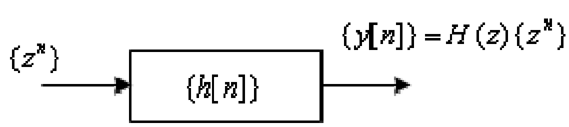


Fig 5.1

The constant $H(z)$ for a specified value of z is the eigenvalue associated with eigenfunction. In the analysis of LTI system, the usefulness of decomposing a more general signal in terms of eigenfunctions can be seen from the following example. Let $\{x[n]\}$ correspond to a linear combination of two exponentials

$$\{x[n]\} = a_1\{z_1^n\} + a_2\{z_2^n\}$$

From the eigenfunction property and superposition property the response $\{y[n]\}$ is given by

$$\{y[n]\} = a_1 H(z_1)\{z_1^n\} + a_2 H(z_2)\{z_2^n\}$$

More generally if

$$\{x[n]\} = \sum_k a_k\{z_k^n\}$$

then

$$\{y[n]\} = \sum_k a_n H(z_n)\{z_k^n\}$$

Thus if input signal can be represented by a linear combination of exponential signals, the output can also be represented by a linear combination of same exponentials, moreover the coefficient of the linear combination in the output is obtained by multiplying, a_k, the coefficient in the input representation by corresponding eigen value $H(z_k)$The procedure outlined above is useful if we can represent a large class of signals in terms of complex exponentials. In this chapter we will consider representation of aperiodic signals in terms of signals.

The Discrete Time Fourier Transform (DTFT)

Here we take the exponential signals to be $\{e^{j\omega n}\}$ where ω is a real number.
The representation is motivated by the Harmonic analysis, but instead of following the historical development of the representation we give directly the defining equation.

Let $\{x[n]\}$ be discrete time signal such that $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ that is $\{x[n]\}$ sequence is absolutely summable.

The sequence $\{x[n]\}$ can be represented by a Fourier integral of the form

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad (5.1)$$

where

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad (5.2)$$

Equation (5.1) and (5.2) give the Fourier representation of the signal. Equation (5.1) is referred as synthesis equation or the inverse discrete time Fourier transform (IDTFT) and equation (5.2)is Fourier transform in the analysis equation.

Fourier transform of a signal in general is a complex valued function, we can write

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \qquad (5.3)$$

where $X_R(e^{j\omega})$ is the real part of $X(e^{j\omega})$ and $X_I(e^{j\omega})$ is imaginary part of the function. We can also use a polar form

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{\angle X(e^{j\omega})} \tag{5.4}$$

where $|X(e^{j\omega})|$ is magnitude and $\angle X(e^{j\omega})$ is the phase of. We also use the term Fourier spectrum or simply, the spectrum to refer to. Thus $|X(e^{j\omega})|$ is called the magnitude spectrum and $\angle X(e^{j\omega})$ is called the phase spectrum.

From equation (5.2) we can see that $X(e^{j\omega})$ is a periodic function with period 2π i.e.. We can interpret (5.1) as Fourier coefficients in the representation of a perotic function. In the Fourier series analysis our attention is on the periodic function, here we are concerned with the representation of the signal. So the roles of the two equation are interchanged compared to the Fourier series analysis of periodic signals.

Now we show that if we put equation (5.2) in equation (5.1) we indeed get the signal.

Let

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{+j\omega n}d\omega$$

where we have substituted $X(e^{j\omega})$ from (5.2) into equation (5.1) and called the result as.

Since we have used n as index on the left hand side we have used m as the index variable for the sum defining the Fourier transform. Under our assumption that $\{x[n]\}$ sequence is absolutely summable we can interchange the order of integration and summation. Thus

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{+j\omega(n-m)}d\omega\right) \tag{5.5}$$

The integral with the parentheses can be evaluated as

if $m = n$ then $e^{j\omega(n-m)} = 1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 1.d\omega = 1$$

and

if $m \neq n$ then $e^{j\omega(n-m)} = \cos\omega(n-m) + j\sin\omega(n-m)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\omega(n-m)d\omega + \frac{j}{2\pi}\int_{-\pi}^{\pi}\sin\omega(n-m)d\omega$$

$$= \frac{1}{2\pi}\frac{\sin\omega(n-m)}{(n-m)}\Bigg|_{-\pi}^{\pi} \frac{j}{2\pi}\frac{\cos\omega(n-m)}{n-m}\Bigg|_{-\pi}^{\pi}$$

$$= 0$$

Thus in equation (5.5) there is only one non-zero term in RHS, corresponding to $m = n$, and we get. This result is true for all values of n and so equation (5.1) is indeed a representation of signal $\{x[n]\}$ in terms eigenfunctions $\{e^{j\omega n}\}$

In above demonstration we have assumed that $\{x[n]\}$ is absolutely summable. Determining the class of signals which can be represented by equation (5.1) is equivalent to considering the convergence of the infinite sum in equation (5.2). If we fix a value of $\omega = \omega_0$ then, RHS of equation (5.2) is a complex valued series, whose partial sum is given by

$$X_N(e^{j\omega_0}) = \sum_{n=-N}^{N} x[n]e^{-j\omega_0 n}$$

The limit as $N \to \infty$ if the partial sum $X_N(e^{j\omega_0})$ exists if the series is absolutely summable.

$$|X_N(e^{j\omega_0})| : |\sum_{n=-N}^{N} x[n]e^{-j\omega_0 n}|$$

$$: \sum_{n=-N}^{N} |x[n]e^{-j\omega_0 n}|$$

by triangle inequality

$$: \sum_{n=-N}^{N} |x[n]|$$

Since the limit $N \to \infty$ $\sum_{n=-N}^{N} |x[n]|$ exists by our assumption the limit $N \to \infty$ $X_N(e^{j\omega_0})$ exists for every. Furthermore it can be shown that the series converges uniformly to a continuous function of.

If a sequence has only finitely many non-zero terms then it is absolutely summable and so the Fourier transform exists. Since a stable sequence is by definition, an absolutely summable sequence, its Fourier transform also exits.

Example: Let $\{x[n]\} = \{a^n u[n]\}$

Fourier transform of this sequence will exist if it is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n$$

This is a geometric series and sum exists if $|a| < 1$, in that case

$$\sum |a|^n = \frac{1}{1-|a|} < +\infty$$

Thus the Fourier transform of the sequence $\{a^n u[n]\}$ exists if. The Fourier transform is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \tag{5.6}$$

Where the last equality follows from sum of a geometric series, which exists if $|ae^{-j\omega n}| < 1$ i.e.. Absolute summability is a sufficient condition for the existence of a Fourier transform. Fourier transform also exists for square summable sequence.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

For such signals the convergence is not uniform. This has implications in the design of discrete system for filtering.

We also deal with signals that are neither so absolutely summable nor square summable. To deal with some of these signals we allow impulse functions, which is not an ordinary function but a generalized function as a Fourier transform. The impulse function is defined by the following properties

(a) $\int_{-\infty}^{\infty} \delta(\omega) d\omega = 1$

(b) $\int_{-\infty}^{\infty} X(e^{j\omega})\delta(\omega - \omega_0) d\omega = X(e^{j\omega_0})$ if $X(e^{j\omega})$ is continuous at $\omega = \omega_0$;(shifting or convolution property)

(c) $X(e^{j\omega})\delta(\omega) = X(e^{j0})\delta(\omega)$ if $X(e^{j\omega})$ is continuous at $\omega = 0$

Since $X(e^{j\omega})$ is a periodic function, let us consider

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k) \tag{5.7}$$

If we substitute this in equation (5.1) we get

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{h=-\infty}^{\infty} \delta(\omega + 2\pi h) e^{j\omega n} d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega$$

Since there is only one impulse in the interval of integration. Thus we can say that (5.7) represents Fourier transform of a signal such that $x[n] = 1$ for all.

As a generalization of the above example consider a sequence $\{x[n]\}$ whose Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi k), \;\; -\pi < \omega_0 \le \pi$$

substituting this in equation (5.1) we get

$$x[n] \;= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{h=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi h) e^{j\omega n} d\omega \qquad (5.8)$$

$$= \int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega n} d\omega$$

as only one term corresponding to $k = 0$ will be there in the interval of the integration $x[n] = e^{j\omega_0 n}$

So the signal is $\{e^{j\omega_0 n}\}$ when Fourier transform is given by (5.8). More generally if x[n] is sum of an arbitrary set if complex exponentials

$$\{x[n]\} = a_1\{e^{j\omega_1 n}\} + a_2\{e^{j\omega_2 n}\} + ... + a_m\{e^{j\omega_m n}\}$$

Thus its Fourier transform is

$$X(e^{j\omega}) = a_1 \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_1 + 2\pi k) + a_2 \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_2 + 2\pi k) + ... + a_m \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_m + 2\pi k) \qquad (5.9)$$

Thus $X(e^{j\omega})$ is a periodic impulse train, with impulses located at the frequencies $\omega_1, \omega_2 ... \omega_m$ of each of the complex exponentials and at all points that are multiples of 2π from these frequencies. An interval of 2π contains exactly one impulse from each of the summation in RHS of (5.9)

Example: Let $\{x[n]\} = \{\cos \omega_0 n\}$

$$x[n] = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

Hence

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi[\delta(\omega - \omega_0 + 2\pi k) + \delta(\omega + \omega_0 + 2\pi k)]$$

Properties of the Discrete Time Fourier Transform:

In this section we use the following notation. Let $\{x[n]\}$ and $\{y[n]\}$ be two signal, then their DTFT is denoted by $X(e^{j\omega})$ and. The notation

$$\{x[n]\} \leftrightarrow X(e^{j\omega})$$

is used to say that left hand side is the signal x[n] whose DTFT is $X(e^{j\omega})$ is given at right hand side.

1. Periodicity of the DTFT

As noted earlier that the DTFT $X(e^{j\omega})$ is a periodic function of ω with period. This property is different from the continuous time Fourier transform of a signal.

2. Linearity of the DTFT:

If $\{x[n]\} \leftrightarrow X(e^{j\omega})$

and $\{y[n]\} \leftrightarrow Y(e^{j\omega})$

then $a\{x[n]\} + b\{y[n]\} \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$

This follows easily from the defining equation (5.2).

3. Conjugation of the signal:

If $\{x[n]\} \leftrightarrow X(e^{j\omega})$

then $\{x^*[n]\} \leftrightarrow X^*(e^{-j\omega})$

where * denotes the complex conjugate. We have DTFT of $\{x^*[n]\}$

$$\sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} [x[n]e^{j\omega n}]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x[n]e^{-j(-\omega)n}\right]^*$$

$$= X^*(e^{-j\omega})$$

4. Time Reversal

$$\{x[-n]\} \leftrightarrow X(e^{-j\omega})$$

The DTFT of the time reversal sequence is

$$\sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}$$

Let us change the index of summation as $m = -n$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{j\omega m} = X(e^{-j\omega})$$

5. Symmetry properties of the Fourier Transform:

If x[n] is real valued than

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

This follows from property 3. If x[n] is real valued then $x[n] = x^*[n]$, so $\{x[n]\} = \{x^*[n]\}$ and hence

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

expressing $X(e^{j\omega})$ in real and imaginary parts we see that

$$X_R(e^{j\omega}) + jX_I(e^{j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{-j\omega})$$

which implies

$$X_R(e^{j\omega}) = X_R(e^{-j\omega})$$

and

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$$

That is real part of the Fourier transform is an even function of ω and imaginary part is an odd function of.

The magnitude spectrum is given by

$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})} = \sqrt{X_R^2(e^{-j\omega}) + X_I^2(e^{-j\omega})} = |X(e^{-j\omega})|$$

Hence magnitude spectrum of a real signal is an even function of.

The phase spectrum is given by

$$\angle X(e^{j\omega}) = tan^{-1}\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}$$

$$= tan^{-1}\frac{-X_I(e^{-j\omega})}{X_R(e^{-j\omega})}$$

$$= -tan^{-1}\frac{X_I(e^{-j\omega})}{X_R(e^{-j\omega})}$$

$$= -\angle X(e^{-j\omega})$$

Thus the phase spectrum is an odd function of. We denote the symmetric and antisymmetric part of a function by

$$Ev(\{x[n]\}) = \frac{1}{2}\{x[n]\} + \frac{1}{2}\{x^*[-n]\}$$

$$Od(\{x[n]\}) = \frac{1}{2}\{x[n]\} - \frac{1}{2}\{x^*[-n]\}$$

$$Ev(X(e^{j\omega})) = \frac{1}{2}X(e^{j\omega}) + \frac{1}{2}X^*(e^{-j\omega})$$

$$Od(X(e^{j\omega})) = \frac{1}{2}X(e^{j\omega}) - \frac{1}{2}X^*(e^{-j\omega})$$

Then using property (2) and (3) we see that

$$Ev(\{x[n]\}) \leftrightarrow Re\, X(e^{j\omega})$$

$$Od(\{x[n]\}) \leftrightarrow jIm\, X(e^{j\omega})$$

and using property (2) and (4) we can see that

$$Re(\{x[n]\}) \leftrightarrow Ev(X(e^{j\omega}))$$

$$Im(\{x[n]\}) \leftrightarrow Od(X(e^{j\omega}))$$

6. Time shifting and frequency shifting:

$$\{x[n-n_0]\} \leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

$$\{e^{j\omega_0 n} x[n]\} \leftrightarrow X(e^{j(\omega-\omega_0)})$$

These can be proved very easily by direct substitution of $x[n-n_0]$ in equation(5.2) and $X(e^{j(\omega-\omega_0)})$ in equation (5.1).

7. Differencing and summation:

$$\{x[n] - x[n-1]\} \leftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$$

This follows directly from linearity property 2.

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

Consider next the signal $\{y[n]\}$ defined by

since $y[n] - y[n-1] = x[n]$, we are tempted to conclude that the DTFT of $\{y[n]\}$ is DTFT of $\{x[n]\}$

divided by. This is not entirely true as it ignores the possibility of a dc or average term that can result from summation. The precise relationship is

$$\left\{\sum_{m=-\infty}^{n} x[m]\right\} \leftrightarrow \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$$

We omit the proof of this property.

If we take $\{x[n]\} = \{\delta[n]\}$ then we get

$$\{u[n]\} = \{\sum_{m=-\infty}^{n} \delta[n]\} \leftrightarrow \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$$

8. Time and frequency scaling:

For continuous time signals we know that the Fourier transform of $x(at)$ is given by. However if we define a signal $\{x[an]\}$ we run into difficulty as the index an must be an integer. Thus if a is an integer say $a = k > 0$, then we get signal. This consists of taking k^{th} sample of the original signal. Thus the DTFT of this signal looks similar to the Fourier transform of a sampled signal. The result that resembles the continuous time signal is obtained if we define a signal $\{x_{(k)}[n]\}$ by

$$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}], & \text{if n is multiple of k} \\ 0, \text{if n is not a multiple of k} \end{cases}$$

For example $\{x_{(2)}[n]\}$ **is illustrated below**

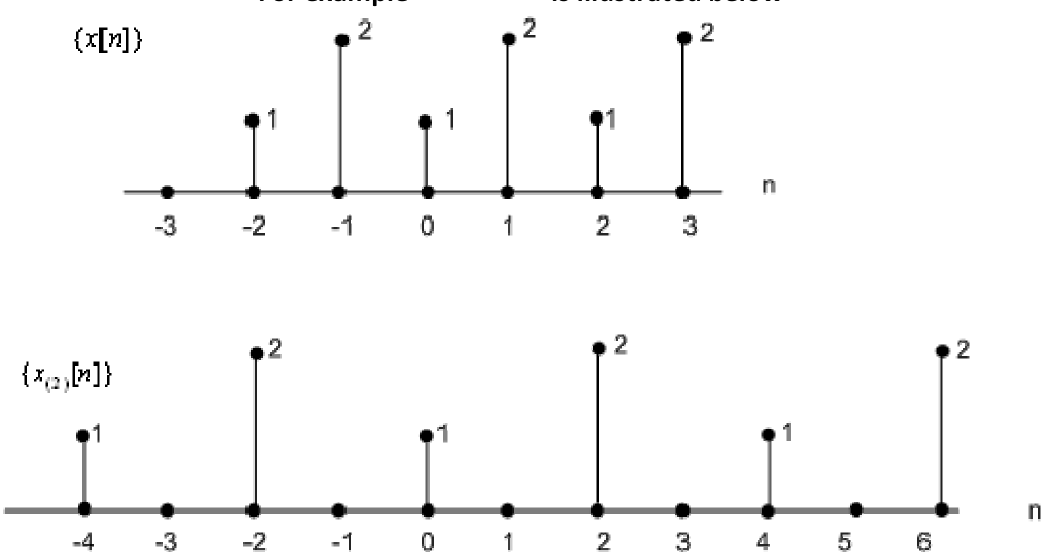


Fig 5.2

The signal $\{x_{(h)}[h]\}$ is obtained by inserting $(k-1)$ zeroes between successive value if signal.

$$X_{(k)}(e^{j\omega}) : \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n}$$

$$: \sum_{m=-\infty}^{\infty} x[km]e^{-j\omega km}$$

$$: X(e^{jk\omega})$$

Here we can note the time frequency uncertainly. Since $\{x_{(k)}[n]\}$ is expanded sequence, the Fourier transform is compressed.

9. Diffentiation in frequency domain

$$X(e^{e^{j\omega}}) \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Differentiating both sides with respect to ω, we obtain

$$\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$$

multiplying both sides by j we obtain

$$\{nx[n]\} \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

10. Passeval's relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega$$

We have

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega n}d\omega\right]^*$$

interchanging summation and integration we get

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})X(e^{j\omega})d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

11. Convolution property:

This is the eigenfunction property of the complex exponential mentioned in the beginning of the chapter. The fourier syntaxis equation (5.1) for the x[n] can be interpreted as a representation of { x[n]} in terms of linear combinations of complex exponential with amplitude proportional to. Each of these complex exponential is an eigenfunction of the LTI system and so the amplitude $Y(e^{j\omega})$ in the decomposition of $\{y[n]\}$ will be $X(e^{j\omega})H(e^{j\omega})$, where $H(e^{j\omega})$ is the Fourier transform of the impulse response. We prove this formally. The output $\{y[n]\}$ is given is terms of convolution sum, so

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum {}^{\infty}{}_{k=-\infty} h[k]x[n-k]\right) e^{-j\omega n}$$

interchanging order of the summation

$$= \sum_{l=-\infty}^{\infty} h[h] \sum_{n=-\infty}^{\infty} x[n-k]e^{-j\omega n}$$

Let $m = n - k$ then $n = m + k$ and we get

$$= \sum_{k=-\infty}^{\infty} h[k] \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}$$

$$= H(e^{j\omega})X(e^{j\omega})$$

Thus if $\{y[n]\} = \{h[n]\} \star \{x[n]\}$

then

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \tag{5.20}$$

convolution in time domain becomes multiplication in the frequency domain. The fourier transform of the impulse response $\{h[n]\}$ is known as frequency response of the system.

12. The Modulation or windowing property

Let us find the DTFT of product of two sequences

$$\{x[n]\}\{y[n]\} = \{x[n]y[n]\} = \{z[n]\}$$

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-j\omega n}$$

Substituting for x[n] in terms of IDFT we get

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\alpha})e^{j\alpha n} d\alpha \right) y[n]e^{-j\omega n}$$

interchanging order of integration and summation

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\alpha}) \left[\sum_{n=-\infty}^{\infty} y[n]e^{-j(\omega-\alpha)n} \right] d\alpha$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\alpha})Y(e^{j(\omega-\alpha)})d\alpha$$

This looks like convolution of two functions, only the interval of integration is $-\pi$ to. $X(e^{j\omega})$ and $Y(e^{j\omega})$ one periodic functions, and equation (5.21) is called periodic convolution. Thus

$$\{x[x[n]y[n]\} \longleftrightarrow \frac{1}{2\pi} X(e^{j\omega}) \otimes Y(e^{j\omega})$$

where $\otimes$ denotes periodic convolution.
We summarize these properties in Table (5.1)

Table 5.1: Properties of Discrete time Fourier Transform

Aperiodic signal	Discrete time fourier transform
{x[n]}	$X(e^{j\omega})$

$\{y[n]\}$	$Y(e^{j\omega})$
$a\{x[n]\}+b\{y[n]\}$	$aX(e^{j\omega})+bY(e^{j\omega})$
$\{x[n-n_0]\}$	$e^{-j\omega n_0}X(j\omega)$
$\{e^{-j\omega n_0}x[n]\}$	$X(e^{j(\omega-\omega n_0})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$\{x[n]\}*\{y[n]\}$	$X(e^{j\omega})Y(e^{j\omega})$
$\{x[n]y[n]\}$	$\frac{1}{2\pi}[X(e^{j\omega})\otimes Y(e^{j\omega})]$
$\{x[n]-x[n-1]\}$	$(1-e^{-j\omega})X(e^{j\omega})$
$\{\sum_{k=-\infty}^{\infty} x[n]\}$	$\frac{1}{1-e^{-j\omega}}X(e^{-j\omega})+\pi X(e^{-j0})\sum -k=-\infty^{\infty}\delta(\omega+2\pi k)$
$\{nx[n]\}$	$\frac{jdX(e^{j\omega})}{d\omega}$
$\sum_{n=-\infty}^{\infty} x[n]y^*[n]$	$\frac{1}{2\pi}\int -\pi^{\pi}X(e^{j\alpha})Y^*(e^{j(\omega-\alpha)})d\omega$

The frequency response of systems characterized by linear constant coefficient difference equation.

As we have seen earlier, constant coefficient linear difference equation with zero initial condition can be used to describe some linear time invariant systems.

The input-output $\{x[n]\}$ and $\{y[n]\}$ are related by

$$\sum_{k=0}^{0} a_k y[n-k]=\sum_{k=0}^{n} x[n-k] \qquad (5.22)$$

We assume that Fourier transforms of $\{x[n]\},\{y[n]\}$ and $\{h[n]\}$, ($\{h[h]\}$ is the impulse response of the system) exist, then convolution property implies that

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Taking fourier transform of both sides of equation (5.22) and using linearity and time shifting property of the Fourier transform we get

$$\sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} X(e^{j\omega})$$

or

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \qquad (5.23)$$

Thus we see that the frequency response is ratio of polynomials in the variable. The numerator coefficients are the coefficients of $x[n-k]$ in equation (5.22) and denominator coefficients are the coefficients of $y[n-k]$ in equation (5.22). Thus we can write the frequency response by inspection.

Example 2: Consider an LTI system initially at rest described by the difference equation

$$y[n] - ay[n-1] = x[n], \quad |a| < 1$$

The frequency response of the system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

We can use the inverse fourier transform to get the impulse response

$$h[n] = a^n u[n]$$

Discrete Fourier series Representation of a periodic signal

Suppose that $\tilde{x}[n]$ is a periodic signal with period N, that is

$$\tilde{x}[n+N] = \tilde{x}[n]$$

As is continues time periodic signal, we would like to represent $\tilde{x}[n]$ in terms of discrete time complex exponential signals are given by

$$e^{\frac{j2\pi}{N}kn}, \; k = 0, \pm 1, \pm 2, \ldots \qquad (6.1)$$

All these signals have frequencies is that are multiples of the some fundamental frequency, $\frac{2\pi}{n}$, and thus harmonically related.

These are two important distinction between continuous time and discrete time complex exponential.

The first one is that harmonically related continuous time complex exponential $e^{j\Omega_0 kt}$ are all distinct for different values of *k* , while there are only *N* different signals in the set.
The reason for this is that discrete time complex exponentials which differ in frequency by integer multiple of 2π are identical. Thus

$$\{e^{j\frac{2\pi}{N}kn}\} = \{e^{j\frac{2\pi}{N}(k+N)n}\}$$

So if two values of *k* differ by multiple of *N* , they represent the same signal. Another difference between continuous time and discrete time complex exponential is that $\{e^{j\Omega_0 kt}\}$ for different *k* have period $\dfrac{2\pi}{\Omega_0 |k|}$ which changes with *k*. In discrete time exponential, if *k* and *N* are relative prime than the period is *N* and not *N/k.* Thus if *N* is a prime number, all the complex exponentials given by (6.1) will have period *N*. In a manner analogous to the continuous time, we represent the periodic signal $\tilde{x}[n]$ as

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} \tag{6.2}$$

where

$$\tilde{X}[k] = \sum_{n-0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn} \tag{6.3}$$

In equation (6.2) and (6.3) we can sum over any consecutive *N* values. The equation (6.2) is synthesis equation and equation (6.3) is analysis equation. Some people use the faction 1 */N* in analysis equation. From (6.3) we can see easily that

$$\tilde{x}[k] = \tilde{x}[k+N]$$

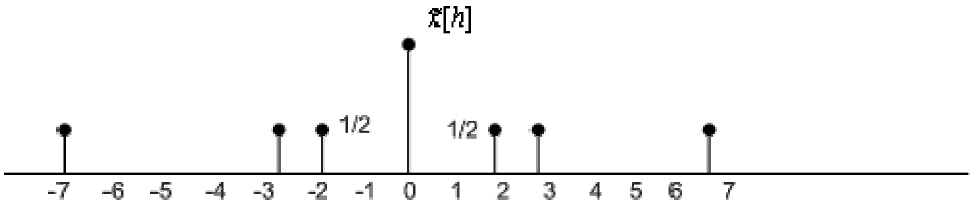
Thus discrete Fourier series coefficients are also periodic with the same period *N*.

Example 1:

$$\{\tilde{x}[n]\} = \{\cos\frac{4\pi}{5}\},$$

$$\tilde{x}[n] = \frac{1}{2}\left(e^{j\frac{2\pi}{5}.2} + e^{-j\frac{2\pi}{5}.2}\right)$$

So, $\tilde{X}[2] = \dfrac{5}{2}$ and $\tilde{X}[-2] = \dfrac{5}{2}$, since the signal is periodic with periodic with period 5, coefficients are also periodic with period 5, and.



Now we show that substituting equation (6.3) into (6.2) we indeed get.

$$\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} \frac{1}{N} \sum_{m=0}^{N-1} \tilde{x}[m] e^{-j\frac{2\pi}{N}km} e^{j\frac{2\pi}{N}kn}$$

interchanging the order of summation we get

$$= \sum_{m=0}^{N-1} \tilde{x}[m] \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)k} \tag{6.4}$$

Now the sum

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)k} = 1$$

if $n - m$ multiple of N
and for ($n - m$) not a multiple of N this is a geometric series, so sum is

$$\frac{1}{N}\left(\frac{1-e^{j\frac{2\pi}{N}(m-n)N}}{1-e^{j\frac{2\pi}{N}(m-n)}}\right) = 0$$

As m varies from 0 to N - 1, we have only one value of m namely $m = n$, for which the inner sum if non-zero. So we set the RHS of (6.4) as.

Properties of Discrete-Time Fourier Series

Here we use the notation similar to last chapter. Let $\{\tilde{x}[n]\}$ be periodic with period N and discrete Fourier series coefficients be $\{\xi\}$ then the write

$$\{\tilde{x}[n]\} \leftrightarrow \tilde{X}[k]$$

where LHS represents the signal and RHS its DFS coefficients

1. Periodicity DFS coefficients:

As we have noted earlier that DFS Coefficients $\{\tilde{X}[k]\}$ are periodic with period *N.*

2. Linearity of DFS:

If

$$\{\tilde{x}[n]\} \leftrightarrow \{\tilde{X}[k]\}$$

$$\{\tilde{y}[n]\} \leftrightarrow \{\tilde{Y}[k]\}$$

If both the signals are periodic with same period N then

$$A\{\tilde{x}[n]\} + B\{\tilde{y}[n]\} \leftrightarrow A\{\tilde{X}[k]\}B\{\tilde{Y}[k]\}$$

3. Shift of a sequence:

$$\{\tilde{x}[n-m]\} \leftrightarrow \left\{ e^{-j\frac{2\pi}{N}mk} \tilde{X}[k] \right\} \tag{6.5}$$

$$\left\{ e^{j\frac{2\pi}{N}ln} \tilde{x}[n] \right\} \leftrightarrow \{\tilde{X}[k-l]\} \tag{6.6}$$

To prove the first equation we use equation (6.3). The DFS coefficients are given by

$$\sum_{n=0}^{N-1} \tilde{x}[n-m] e^{-j\frac{2\pi}{N}kn}$$

let $n - m = l$, we get

$$= \sum_{l-m}^{N-1-m} \tilde{x}[l] e^{-j\frac{2\pi}{N}k(m+l)}$$

since $\tilde{x}[l]$ is periodic we can use any N consecutive values, then

$$= e^{-j\frac{2\pi}{N}km} \sum_{l=0}^{N-1} \tilde{x}[l] e^{-j\frac{2\pi}{N}kl}$$

$$= e^{-j\frac{2\pi}{N}km} \tilde{X}[n]$$

We can prove the relation (6.6) in a similar manner starting from equation (6.3)

4. Duality:

From equation (6.2) and (6.3) we can see that synthesis and analysis equation differ only in sign of the exponential and factor 1/N. If $\{\tilde{x}[n]\}$ is periodic with period N , then $\{\tilde{X}[k]\}$ is also periodic with period N. So we can find the discrete fourier series coefficients of $\tilde{X}[n]$ sequence.

From equation (6.2) we see that

$$N\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}$$

Thus

$$N\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k] e^{-j\frac{2\pi}{N}kn}$$

Interchanging the role of k and n we get

$$N\tilde{x}[-k]=\sum_{k=0}^{N-1}\tilde{X}[n]e^{-j\frac{2\pi}{N}kn}$$

comparing this with (6.3) we see that DFS coefficients of $\{\tilde{X}[n]\}$ are $\{N\tilde{x}[-k]\}$, the original periodic sequence is reversed in time and multiplied by *N.* This is known as duality property. If

$$\{\tilde{x}[n]\}\leftrightarrow\{\tilde{X}[k]\} \qquad (6.7)$$

then

$$\{\tilde{X}[n]\}\leftrightarrow\{N\tilde{x}[-k]\} \qquad (6.8)$$

5. Complex conjugation of the periodic sequence:

$$\{\tilde{x}^*[n]\}\leftrightarrow\{X^*[-k]\}$$

substituting in equation (6.3) we get

$$\sum_{n=0}^{N-1}\tilde{x}^*[n]e^{-j\frac{2\pi}{N}kn}=\left[\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}(-k)n}\right]^*$$

6. Time reversal:

$$\{\tilde{x}[-n]\}\leftrightarrow\{\tilde{X}[-k]\}$$

From equation (6.3) we have the DFS coefficient

$$\sum_{n-0}^{N-1}\tilde{x}[-n]e^{-j\frac{2\pi}{N}kn}$$

putting *m* = - *n* we get

$$=\sum_{m=-(N-1)}^{0}\tilde{x}[m]e^{j\frac{2\pi}{N}km}$$

Since $\tilde{x}[m]$ is periodic, we can use any *N* consecutive values

$$=\sum_{m=0}^{N-1}\tilde{x}[m]e^{j\frac{2\pi}{N}km}$$

$$=\tilde{X}[-k]$$

7. Symmetry properties of DFS coefficient:

In the last chapter we discussed some symmetry properties of the discrete time Fourier transform of aperiodic sequence. The same symmetry properties also hold for DFS coefficients and their derivation is also similar in style using linearity, conjugation and time reversal properties DFS coefficients.

8. Time scaling:

Let us define

$$\tilde{x}_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is multiple of } m \\ 0, \text{if } n \text{ is not a multiple of } m \end{cases}$$

sequence $\{\tilde{x}_{(m)}[n]\}$ is obtained by inserting (m - 1) zeros between two consecutive values of. Thus Thus $\{\tilde{x}_{(m)}[n]\}$ is also periodic, but period is mN. The DFS coefficients are given by

$$\sum_{n=0}^{mN-1} \tilde{x}_{(m)}[n] e^{-j\frac{2\pi}{mN}kn}$$

putting $n - lm + r,\ 0 \le l \le N-1,\ 0 \le r < m$

$$= \sum_{l=0}^{N-1} \tilde{x}[l] e^{-j\frac{2\pi}{N}\frac{k}{m}(lm)}$$

as non zero terms occur only when r = 0

$$= \tilde{x}[h].$$

If we define $\tilde{y}[n] = \tilde{x}[nM]$ then $\tilde{y}[n]$ is periodic with period equal to least common multiple (LCM) of M and N. The relationship between DFS coefficients is not simple and we omit it here.

9. Difference

$$\{(\tilde{x}[n] - \tilde{x}[n-1])\} \leftrightarrow \left\{\left(1 - e^{-j\frac{2\pi}{N}kn}\right)\tilde{X}[k]\right\}$$

This follows from linearity property.

10. Accumulation

Let us define

$$\tilde{y}[n] = \sum_{k=-\infty}^{n} \tilde{x}[k]$$

$\{\tilde{y}[n]\}$ will be bounded and periodic only if the sum of terms of $\tilde{x}[n]$ over one period is zero, i.e. $\sum_{n=0}^{N-1} \tilde{x}[n] = 0$, which is equivalent to. Assuming this to be true

$$\left\{\sum_{k=-\infty}^{n} \tilde{x}[k]\right\} \leftrightarrow \left\{\left(\frac{1}{1 - e^{-j\frac{2\pi k}{N}}}\right)\tilde{X}[h]\right\}$$

11. Periodic convolution

Let $\{\tilde{x}_1[n]\}$ and $\{\tilde{x}_2[n]\}$ be two periodic signals having same period N with discrete Fourier series coefficients denoted by $\{\tilde{X}_1[k]\}$ and $\{\tilde{X}_2[k]\}$ respectively. If we form the product $\tilde{x}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k]$ then we want to find out the sequence $\{\tilde{x}_3[n]\}$ whose DFS coefficients are. From the synthesis equation we have

$$\tilde{x}_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}_3[k] e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}_1[k]\tilde{X}_2[k] e^{j\frac{2\pi}{N}kn}$$

substituting for $\tilde{X}_1[k]$ in terms of $\tilde{x}n$ we get

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} \tilde{x}_1[m] e^{-j\frac{2\pi}{N}km} \tilde{X}_2[k] e^{j\frac{2\pi}{N}kn}$$

interchanging order of summations we get

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}_2[k] e^{j\frac{2\pi}{N}(n-m)k}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \qquad (6.15)$$

as inner sum can be recognized as $\tilde{x}_2[n-m]$ from the synthesis equation. Thus

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \leftrightarrow \{\tilde{X}_1[k]\tilde{X}_2[k]\}$$

The sum in the equation (6.15) looks like convolution sum, except that the summation is over one period. This is known as periodic convolution. The resulting sequence $\{\tilde{x}_3[n]\}$ is also periodic with period *N.* This can be seen from equation (6.15) by putting *m* + *N* instead of *m.*

The Duality theorem gives analogous result when we multiply two periodic sequences.

$$\{\tilde{x}_1[n]\tilde{x}_2[n]\} \leftrightarrow \left\{\frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}_1[l]\tilde{X}_2[k-l]\right\}$$

The DFS coefficients are obtained by doing periodic convolution of $\{\tilde{X}_1[k]\}$ and $\{\tilde{X}_2[k]\}$ and multiplying the result by 1/N. We can also prove this result directly by starting from the analysis equation. The periodic convolution has properties similar to the aperiodic (linear convolution).It is cumulative, associative and distributes over additions of two signals.

The properties of DFS representation of periodic sequence are summarized in the Table 6.1

	Periodic sequence (period N)	DFS coefficients (Period N)
1.	$\{\tilde{x}[n]\}$	$\{\tilde{x}[k]\}$ period N
2.	$a\{\tilde{x}[n]\}+b\{\tilde{y}[n]\}$	$a\{\tilde{X}[k]\}+b\{\tilde{Y}[k]\}$
3.	$\{\tilde{x}[n]\}$	$N\{X[-k]\}$
4.	$\{\tilde{x}[n-m]\}$	$\{e^{-j\frac{2\pi}{N}km}\tilde{X}[k]\}$
5.	$\{e^{j\frac{2\pi}{N}ln}\tilde{x}[n]\}$	$\{X[k-1]\}$
6.	$\{\tilde{x}^{*}[n]\}$	$\{\tilde{X}^{*}[-k]\}$
7.	$\{x[-n]\}$	$\{\tilde{X}[-k]\}$
8.	$\tilde{x}_{(m)}[n]=\begin{cases} x\left[\frac{m}{m}\right], & n=lm \\ 0, & \text{otherwise}\end{cases}$ (periodic with period mN)	$\{\tilde{X}[k]\}$, (viewed as periodic with period mN)
9.	$\{x[n]-x[n-1]\}$	$\{(1-e^{-j\frac{2\pi}{N}h})\tilde{X}[k]\}$
10.	$\left\{\sum_{m=-\infty}^{n}\tilde{x}[m]\right\}$ (periodic only if $\tilde{x}[0]=0$)	$\left\{\frac{1}{1-e^{-j\frac{2\pi}{N}k}}\tilde{X}[k]\right\}$
11.	$\left\{\sum_{m=0}^{N-1}\tilde{x}[m]\tilde{y}[n-m]\right\}$	$\{\tilde{X}[k]\tilde{Y}[k]\}$
12.	$\{\tilde{x}[n]\tilde{y}[n]\}$	$\left\{\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}[l]\tilde{y}[k-l]\right\}$
13.	$\{Re[\tilde{x}[n]]\}$	$\{\tilde{X}_e[k]\}=\left\{\frac{1}{2}(\tilde{X}[k]+\tilde{X}^{*}[-k])\right\}$
14.	$\{jI_m[\tilde{x}[n]]\}$	$\{\tilde{X}_0[k]\}=\left\{\frac{1}{2}(\tilde{X}[k]-\tilde{X}^{*}[-k])\right\}$
15.	$\{\tilde{x}_e[n]\}=\left\{\frac{1}{2}(\tilde{x}[n]+\tilde{x}^{*}[-n])\right\}$	$\{Re\,(X[n])\}$
16.	$\{\tilde{x}_0[n]\}=\left\{\frac{1}{2}(\tilde{x}[n]-\tilde{x}^{*}[n])\right\}$	$\{jI_m[\tilde{x}[n]]\}$
17.	If $\{\tilde{x}[n]\}$ is real then	$X[k]=X^{*}[-k]$

		$\mathrm{Re}[\tilde{X}[k]] = \mathrm{Re}[\tilde{X}[-k]]$ $\mathrm{Im}[\tilde{X}[k]] = -\mathrm{Im}[\tilde{X}[-k]]$ $\lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert$ $\measuredangle \tilde{X}[k] = -\measuredangle \tilde{X}[-k]$

Table 6.1

Fourier Transform of periodic signals

If $\{\tilde{x}[n]\}$ is periodic with period N, then we can write

$$\{\tilde{x}[n]\} = \left\{\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{-j\frac{2\pi}{N}kn}\right\}$$

Using equation (5.9) we see that

$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N}\sum_{l=-\infty}^{\infty}\sum_{k=0}^{N-1}\tilde{X}[k]\delta\left(\omega - \frac{2\pi}{N}k + 2\pi l\right)$$
$$= \frac{2\pi}{N}\sum_{l=-\infty}^{\infty}\tilde{X}[k]\delta\left(\omega - \frac{2\pi l}{N}\right)$$

as $\tilde{X}[k]$ is periodic with period *N.*

Example:

Consider the periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty}\delta[n - rN]$$

then

$$\tilde{X}[k] = \sum_{n=1}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$$
$$= 1$$

as only one term corresponding to *n* = 0 is non zero. Thus the DTFT is

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right)$$

Fourier Representation of Finite Duration sequence

The Discrete Fourier Transform (DFT)

We now consider the sequence $\{x[n]\}$ such that $x[n] = 0,\ n < 0$ and. Thus $x[n]$ can be take non-zero values only for. Such sequences are known as finite length sequences, and *N* is called the length of the sequence. If a sequence has length *M*, we consider it to be a length *N* sequence where. In these cases last (N - M) sample values are zero. To each finite length sequence of length N we can always associate a periodic sequence $\{\tilde{x}[n]\}$ defined by

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n - mN] \qquad (6.16)$$

Note that $\{\tilde{x}[n]\}$ defined by equation (6.16) will always be a periodic sequence with period N, whether $\{x[n]\}$ is of finite length *N* or not. But when $\{x[n]\}$ has finite length N, we can recover the sequence $\{x[n]\}$ from $\{\tilde{x}[n]\}$ by defining

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \qquad (6.17)$$

This is because of $\{x[n]\}$ has finite length *N* , then there is no overlap between terms $x[n]$ and $x[n - mN]$ for different values of.

Recall that if n = *kN* + r, where $0 \le r \le N-1$ then *n* modulo *N* = *r* ,

i.e. we add or subtract multiple of *N* from *n* until we get a number lying between 0 to *N* - 1. We will use ((*n*))*N* to denote *n* modulo *N.* Then for finite length sequences of length *N* equation (6.16) can be written as

$$\tilde{x}[n] = x[((n))_N] \qquad (6.18)$$

We can extract $\{x[n]\}$ from $\{\tilde{x}[n]\}$ using equation (6.17). Thus there is one-to- one correspondance between finite length sequences $\{x[n]\}$ of length *N* , and periodic sequences $\{\tilde{x}[n]\}$ of period *N.*

Given a finite length sequence $\{x[n]\}$ we can associate a periodic sequence $\{\tilde{x}[n]\}$ with it. This periodic sequence has discrete Fourier series coefficients $\{\tilde{X}[k]\}$ which are also periodic with period *N.*From equations (6.2) and (6.3) we see that we need values of $\{\tilde{x}[n]\}$ for $0 \le k \le N-1$ and $\tilde{X}[k]$ for 0 = *k* = *N* - 1. Thus we define discrete Fourier transform of finite length sequence $\{x[n]\}$ as

$$\tilde{X}[k] = \begin{cases} \tilde{X}[k], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

where $\{\tilde{X}[h]\}$ is DFS coefficient of associated periodic sequence. From $\{X[k]\}$ we can get $\{\tilde{X}[h]\}$ by the relation.

$$\tilde{X}[k] = X[((k))_N] = X[(k \bmod ulo\ N)]$$

then from this we can get $\{\tilde{x}[n]\}$ using synthesis equation (6.2) and finally $\{x[n]\}$ using equation (6.17). In equations (6.2) and (6.3) summation interval is 0 to *N* - 1, we can write *X* [*k*] directly in terms of *x*[*n*], and *x*[*n*] directly in terms of *X*[*k*] as

$$X[k]=\begin{cases}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}, & 0\le k\le N-1\\ 0, & \text{otherwise}\end{cases}$$

$$x[n]=\begin{cases}\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}hn}=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi}{N}kn}, & 0\le N\le N-1\\ 0, & \text{otherwise}\end{cases}$$

For convenience of notation, we use the complex quantity

$$W_N = e^{-j\frac{2\pi}{N}} \tag{6.19}$$

with this notation, DFT analysis and synthesis equations are written a follows

Analysis equation:

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn},\ 0\le k\le N-1 \tag{6.20}$$

Synthesis equation:

$$x[k]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn},\ 0\le n\le N-1 \tag{6.21}$$

If we use values of *k* and *n* outside the interval 0 to N - 1 in equation (6.20) and (6.21), then we will not get values zero, but we will get periodic repetition of $X[k]$ and $x[n]$ respectively. In defining DFT, we are concerned with values only in interval 0 to *N* - 1. Since a sequence of length M can also be considered a sequence of length $N, N\ge M$, we also specify the length of the sequence by saying N-point-DFT, of sequence.

Sampling of the Fourier transform:

For sequence $\{x[n]\}$ of length *N*, we have two kinds of representations, namely, discrete time Fourier transform $X(e^{j\omega})$ and discrete Fourier transform. The DFT values $X[k]$ can be considered as samples of $X(e^{j\omega})$

$$X[k]=\sum_{k=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}=\sum_{n=-\infty}^{\infty}x[n]e^{-j\frac{2\pi}{N}kn}$$

(as x[n] = 0 n < 0, for *n* < 0, and n > N - 1)

$$=X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k} \tag{6.22}$$

Thus is $X[k]$ is obtained by sampling $X(e^{j\omega})$ at.

Properties of the discrete Fourier transform

Since discrete Fourier transform is similar to the discrete Fourier series representation, the properties are similar to DFS representation. We use the notation

$$\{x[n]\} \leftrightarrow \{X[k]\}$$

to say that $\{X[k]\}$ are DFT coefficient of finite length sequence.

1. Linearity

If two finite length sequence have length M and N , we can consider both of them with length greater than or equal to maximum of *M* and *N*. Thus if

$$\{x[n]\} \leftrightarrow \{X[k]\}$$

$$\{y[n]\} \leftrightarrow \{Y[k]\}$$

then

$$a\{x[n]\} + b\{y[n]\} \leftrightarrow a\{X[k]\} + b\{Y[k]\}$$

where all the DFTs are N-point DFT. This property follows directly from the equation (6.20)

2. Circular shift of a sequence

If we shift a finite length sequence $\{x[n]\}$ of length *N* , we face some difficulties. When we shift it in right direction $\{x[n-n_0]\}, n_0 > 0$ the length of the sequence will becam $(N+n_0)$ according to definition. Similarly if we shift it left $\{x[n-n_0]\}, n_0 < 0$*,* if may no longer be a finite length sequence as $x[n+n_0]$ may not be zero for n < 0. Since DFT coefficients are same as DFS coefficients, we define a shift operation which looks like a shift of periodic sequence. From $\{x[n]\}$ we get the periodic sequence $\{\tilde{x}[n]\}$ defined by

$$\tilde{x}[n] = x[((n))_N]$$

We can shift this sequence by m to get

$$\tilde{y}[n] = \tilde{x}[n-m]$$

Now we retain the first *N* values of this sequence

$$y[n] = \begin{cases} \tilde{y}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

This operation is shown in figure below for *m* = 2, *N* = 5.

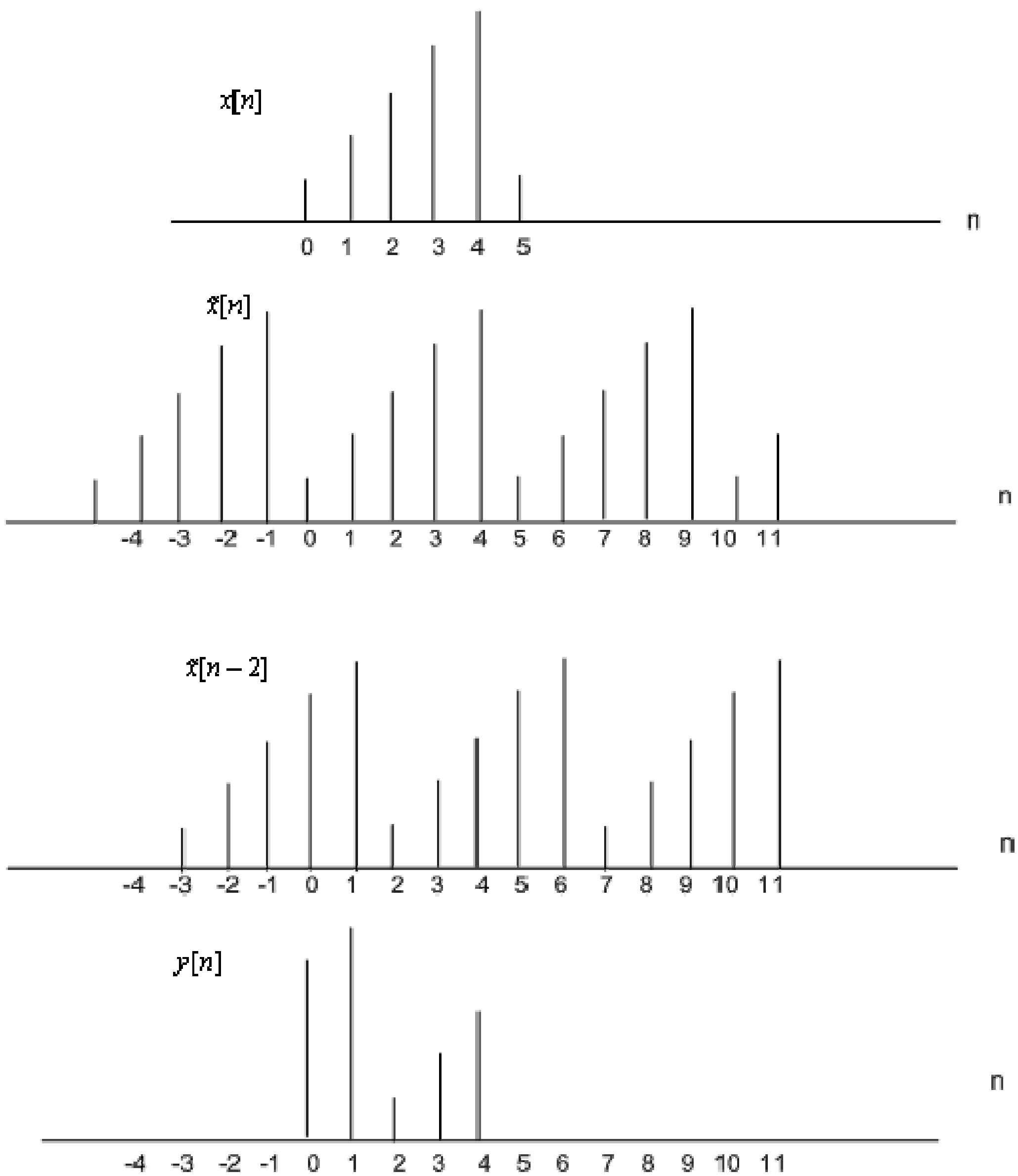


Fig 6.1

We can see that $\{y[n]\}$ is not a shift of sequence. Using the propertiesof the modulo arithmetic we have

$$\tilde{x}[n-m] = x[((n-m))_N]$$

and

$$y[n] = \begin{cases} x[((n-m))], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \qquad (6.23)$$

The shift defined in equation (6.23) is known as circular shift. This is similar to a shift of sequence in a circular register.

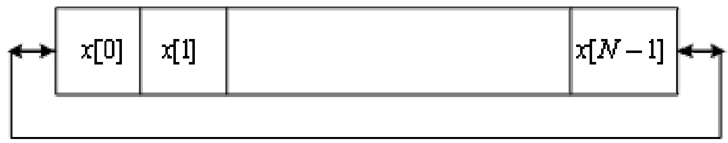


Fig 6.2

3. Shift property of DFT

From the definition of the circular shift, it is clear that it corresponds to linear shift of the associated periodic sequence and so the shift property of the DFS coefficient will hold for the circular shift. Hence

$$\{[((n-m))_N],\, 0 \le n \le N-1\} \leftrightarrow \{W_N^{km}\, X[k]\} \qquad (6.24)$$

and

$$\{W_N^{-nl} x[n]\} \leftrightarrow \{X[((k-l))_N \;\; 0 \le k \le N-1\} \qquad (6.25)$$

4. Duality

We have the duality for the DFS coefficient given by $\{\tilde{X}[n]\} \leftrightarrow \{N\tilde{x}[-k]\}$, retaining one period of the sequences the duality property for the DFT coefficient will become

$$\{\tilde{X}[n]\} \leftrightarrow \{N x[((-k))_N,\; 0 \le k \le N-1\}$$

5. Symmetry properties

We can infer all the symmetry properties of the DFT from the symmetry properties of the associated periodic sequence $\{\tilde{x}[n]\}$ and retaining the first period. Thus we have

$$\{\tilde{x}^*[n]\} \leftrightarrow \{X^*[((-k))_N],\; 0 \le k \le N-1\}$$

and

$$\{X^*[((-n))_N],\; 0 \le n \le N-1\} \leftrightarrow \{X^*[k]\}$$

We define conjugate symmetric and anti-symmetric points in the first period 0 to *N* - 1 by

$$x_{ep}[n] = \tilde{x}_e n = \frac{1}{2}(x[n] + x^*[((-n))_N]),\; 0 \le n \le N-1$$

$$x_{op}[n] = \tilde{x}_o n = \frac{1}{2}(x[n] - x^*[((-n))_N]),\; 0 \le n \le N-1$$

Since

$$((-n))_N = \begin{cases} 0 & n = 0 \\ N-n, & 1 \le n \le N-1 \end{cases}$$

the above equation similar to

$$x_{ep}[n] = \begin{cases} \mathrm{Re}(x[0]), & n = 0 \\ \frac{1}{2}(x[n] - x^*[N-n]), & 1 \le n \le N-1 \end{cases}$$

(6.26)

$$x_{op}[n] = \begin{cases} j\mathrm{Im}(x[0]), & n = 0 \\ \frac{1}{2}(x[n] - x^*[N-n]), & 1 \le n \le N-1 \end{cases}$$

(6.27)

$$x[n] = x_{ep}[n] + x_{op}[n]$$

$\{x_{ep}[n]\}$ and $\{x_{op}[n]\}$ are referred to as periodic conjugate symmetric and periodic conjugate anti-symmetric parts of. In terms if these sequence the symmetric properties are

$$\{\mathrm{Re}(x[n])\} \leftrightarrow \{X_{ep}[k]\}$$

$$\{j\mathrm{Im}(x[n])\} \leftrightarrow \{X_{op}[k]\}$$

$$\{x_{ep}[n]\} \leftrightarrow \{\mathrm{Re}(X[k])\}$$

$$\{x_{op}[n]\} \leftrightarrow \{j\mathrm{Im}(X[k])\}$$

6. Circular convolution

We saw that multiplication of DFS coefficients corresponds of periodic convolution of the sequence. Since DFT coefficients are DFS coefficients in the interval, $0 \le k \le N-1$, they will correspond to DFT of the sequence retained by periodically convolving associated periodic sequences and retaining their first period.

$$\tilde{x}[n] = x[((n))_N]$$

$$\tilde{y}[n] = x[((n))_N]$$

Periodic convolution is given by

$$\tilde{z}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{y}[n-k]$$

using properties of the modulo arithmetic

$$\tilde{z}[n] = \sum_{k=0}^{N-1} x[((k))_N]y[((n-k))_N]$$

and then

$$\tilde{z}[n] = \begin{cases} \tilde{z}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Since $((k))_N = k, \quad 0 \le k \le N-1$ we get

$$z[n] = \sum_{l=0}^{N-1} x[k]y[((n-k))_N], \qquad 0 \le n \le N-1$$

The convolution defined by equation (6.28) is known as N-point-circular convolution of sequence $\{x[n]\}$ and $\{y[n]\}$, where both the sequence are considered sequence of length N. From the periodic convolution property of DFS it is clear that DFT of $\{z[n]\}$ is. If we use the notation $\{x[n]\}\{y[n]\}$ to denote the N point circular convolution we see that

$$\{x[n]\} \quad \{y[n]\} \leftrightarrow \{X[h]Y[k]\} \tag{6.29}$$

In view of the duality property of the DFT we have

$$\{x[n]\,y[n]\} \leftrightarrow \frac{1}{N}\{X[k]\} \ \{Y[k]\} \tag{6.30}$$

Properties of the Discrete Fourier transform are summarized in the table 6.2

	Finite length sequence (length N)	**N-point DFT (length N)**
1.	$\{x[n]\}$	$\{X[k]\}$
2.	$a\{x[n]\}+b\{y[n]\}$	$a\{X[k]\}+b\{Y[k]\}$
3.	$\{X[n]\}$	$N\{X[((-k))_N]\}$
4.	$\{x[((n-m))_N]\}$	$\{W_N^{km}X[k]\}$
5.	$\{W_N^{-ln}x[n]\}$	$\{X[((k-l))_N]\}$
6.	$\{x[n]\} \quad \{y[n]\}$	$\{X[k]Y[k]\}$
7.	$\{x[n]y[n]\}$	$\frac{1}{N}\{X[k]\}\{Y[k]\}$
8.	$\{x^*[n]\}$	$\{X^*[((-k))_N]\}$
9.	$\{x^*[((-n))_N]\}$	$\{X^*[k]\}$
10.	$\{\mathrm{Re}\, x[n]\}$	$\{X_{ep}[k]\}$
11.	$\{j\mathrm{Im}(x[n])\}$	$\{X_{op}[k]\}$
12.	$\{X_{ep}[n]\}$	$\{\mathrm{Re}(X[k])\}$
13.	$\{X_{op}[n]\}$	$\{j\mathrm{Im}(X[k])\}$
14.	If $\{x[n]\}$ is real sequence	$X\{k\} = X^*\{((-k))_N\}$ $\mathrm{Re}(X[k]) = \mathrm{Re}(X[((-k))_N])$ $I_m(X[k]) = -I_m(X[(\lvert -k\rvert)_N])$ $\lvert X[k]\rvert = \lvert X((-k))_N\rvert$ $\measuredangle X[k] = -\measuredangle X[((-k))_N]$

Linear convolution using the Discrete Fourier Transform

Output of a linear time invariant-system is obtained by linear convolution of input signal with the impulse response of the system. If we multiply DFT coefficients, and then take inverse transform we will get circular convolution. From the examples it is clear that result of circular convolution is different from the result of linear convolution of two sequences. But if we modify the two sequence appropriately we can get the result of circular convolution to be same as linear convolution. Our interest in doing linear convolution results form the fact that fast algorithms for computing DFT and IDFT are available. These algorithms will be discussed in a later chapter. Here we show how we can make result of circular convolution same as that of linear convolution.

If we have sequence $\{x[n]\}$ of length L and a sequence $\{y[n]\}$ of length *M* , the sequence $\{z[n]\}$ obtained by linear convolution has length (*L* + *M* - 1).
This can be seen from the definition

$$
\begin{aligned}
Z[n] &= \sum_{k=-\infty}^{\infty} x[k]\, y[n-k] \\
&= \sum_{k=0}^{L-1} x[k]\, y[n-k]
\end{aligned}
\qquad (6.31)
$$

as x[k] = 0 for. For $n < 0, y[n-k] = 0, 0 \le k \le L-1$ hence. Similarly for $n > L+M-2, y[n-k] = 0,\ 0 \le k \le L-1$, so. Hence $z[n]$ is possibly nonzero only for.
Now consider a sequence $\{\omega[n]\}$, DTFT is given by

$$W(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \omega[n] e^{-j\omega n}$$

writing

$n = mN + l, -\infty < m < \infty, 0 \le l \le N-1$

We get

$$W(e^{j\omega}) = \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} \omega[mN+l] e^{-j\omega(mN+l)}$$

If we take

$$\omega = \frac{2\pi}{N} k$$

we see that

$$W\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} W[l+mN] e^{-j\frac{2\pi}{N}l}$$

Comparing this with the DFT equation (6.), we see that

$W\left(e^{j\frac{2\pi}{N}k}\right)$ can be seen as DFT coefficients of a sequence

$$v[l] = \sum_{m=-\infty}^{\infty} \omega[l+mN], \qquad 0 \le l \le N-1 \qquad (6.32)$$

obviously if $\{\omega[n]\}$ has length less then or equal to *N* , then

$$v[l] = \omega[l], \qquad 0 \le l \le N-1$$

However, if the length of $\{W[n]\}$ is greater than $N, v[l]$ may not be equal to $\omega[n]$ for all values of *l.*

The sequence $\{z[n]\}$ in equation (6.31) has the discrete Fourier transform

$$Z(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$$

The N-point DFT of $\{z[n]\}$ sequence is

$$\begin{aligned} Z[k] &= Z\left(e^{j\frac{2\pi}{N}k}\right) \\ &= X\left(e^{j\frac{2\pi}{N}k}\right)Y\left(e^{j\frac{2\pi}{N}k}\right) \\ &= X[k][Y(k) \end{aligned}$$

where $\{X[k]\}$ and $\{Y[k]\}$ are N-point DFTs of $\{x[n]\}$ and $\{y[n]\}$ respectively. The sequence resulting as the inverse DFT of $Z[k]$ is then by equation (6.32).

$$v[n] = \begin{cases} \sum_{m=-\infty}^{\infty} z[n+mN], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

From the circular convolution property of the DFT we have

$$\{v[n]\} = \{x[n]\}\{y[n]\}$$

Thus, the circular convolution of two-finite length sequences can be viewed as linear convolution, followed time aliasing, defined by equation (6.32). If N is greater than or equal to (*L* + *M* - *l*), then there will be no time aliasing as the linear convolution produces a sequence of length (*L* + *M* - *l*). Thus we can use circular convolution for linear convolution by padding sufficient number of zeros at the end of a finite length sequence. We can use DFT algorithm for calculating the circular convolution.

Definition of the Z-transform

We saw earlier that complex exponential of the from $\{e^{jwn}\}$ is an eigen function of for a LTI System. We can generalize this for signals of the form $\{z^n\}$ where, z is a complex number.

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-0}^{\infty} h[k]z^{n-k} \end{aligned}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n$$

$$= H(z) z^n$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-h} \tag{7.1}$$

Thus if the input signal is $\{z^n\}$ then output signal is. For $z = e^{jw}$ w real (i.e for $|z| = 1$), equation (7.1) is same as the discrete-time fourier transform. The $H(z)$ in equation (7.1) is known as the bilateral z-transform of the sequence. We define for any sequence of a sequence $\{x[n]\}$ as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{7.2}$$

where z is a complex variable. Writing z in polar form we get $z = re^{jw}$, where r is magnitude and ω is angle of.

$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n] (re^{jw})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-jwn} \tag{7.3}$$

This shows that $X(re^{jw})$ is Fourier transform of the sequence. When $r = 1$ the z-transform reduces to the Fourier transform of. From equation (7.3) we see that for convergence of z-transform that Fourier transform of the sequence $\{r^{-n} x[n]\}$ converges. This will happen for some r and not for others. The values of z - for which $\sum_{n=-\infty}^{\infty} r^{-n} |x[n]| < \infty$ is called the region of convergence(ROC). If the ROC contains unit circle (i.e. $r = 1$ or equivalently $|z| = 1$ then the Fourier transform also converges. Following examples show that we must specify ROC to completely specify the z-transform.

Example 1: Let $\{x[n]\} = \{a^n u[n]\}$, then

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

This is a geometric series and converges if $|az^{-1}| < 1$ or. Then

$$X[z] = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, |z| > |a| \qquad (7.4)$$

We see that $X(z) = 0$ at $z = 0$, and $1/X((z) = 0$ at. Values of z where $X(z)$ is zero is called zero of $X(z)$ and value of z where $1/X(z)$ is zero is called a pole of. Here we see that ROC consists of a region in Z-plane which lies outside the circle centered at origin and passing through the pole.

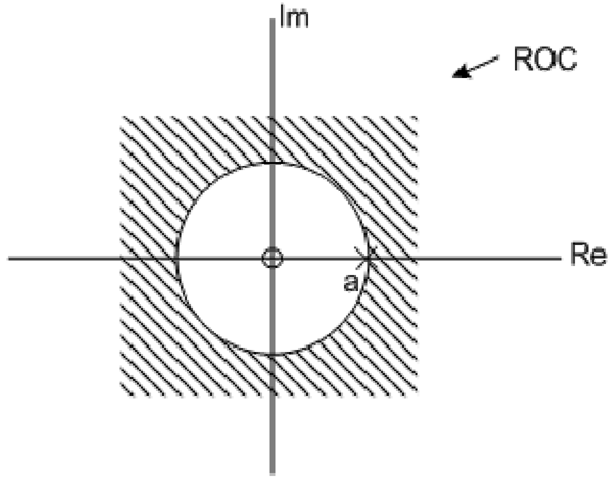


Fig 7.1

Example 2: Let, $\{y[n]\} = \{-a^n u[-n-1]\}$, then

$$Y(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= \sum_{m=1}^{\infty} -a^{-m} z^m$$

This is a geometric series which converges when $|a^{-1} z| < 1$, that is $|z| < |a|$ Then

$$Y(z) = \frac{-a^{-1}z}{1 - a^{-1}z} = -\frac{z}{a - z}$$

$$= \frac{z}{z - a}, \qquad |z| < |a| \qquad (7.5)$$

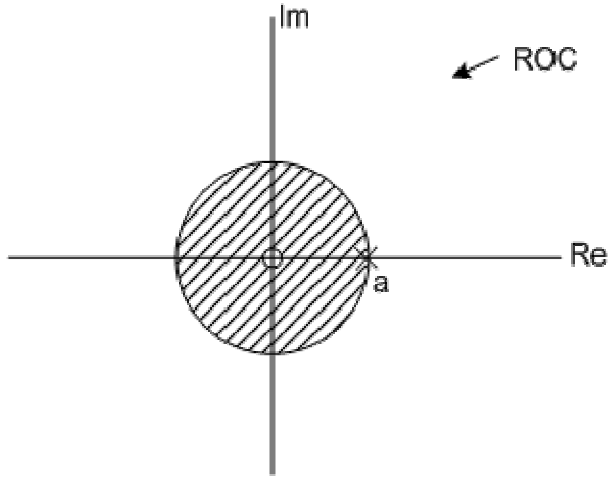


Fig 7.2

Here the ROC is inside the circle of radius. Comparing equation (7.4) and (7.5) we see that algebraic form of $X(z)$ and $Y(z)$ are same, but ROC are different and they correspond to two different sequences. Thus in specifying z-transform, we have to give functional form $X(z)$ and the region of convergence.

Now we state some properties of the region of convergence

Properties of the ROC

1. The ROC of $X(z)$ consists of an annular region in the z-plane, centered about the origin. This property follows from equation (7.3), where we see that convergence depends on r only.
2. The ROC does not certain any poles. Since at poles $X(z)$ does not converge.
3. The ROC is a connected region in z-plane. This property is proved in complex analysis.
4. If $\{x[n]\}$ is a right sided sequence, i.e. $x[n]=0$, for $n < n_0$, and if the circle $Z|=r_0$ is in the ROC, then all finite values of z, for which $|z|>r_0$ will also be in the ROC. For a right sided sequence

$$X(z)=\sum_{n=n_0}^{\infty} x[n]z^{-n}$$

If n_0 is negative then we can write

$$X(z)=\sum_{n=n_0}^{0} x[n]z^{-n}+\sum_{n=1}^{\infty} x[n]z^{-n}$$

Let $Z=re^{jw}$, with $r>r_0$, then, $X(z)$ exists if

$$\sum_{n=n_0}^{-1} |x[n]|r^{-n} + \sum_{n=0}^{\infty} |x[n]|r^{-n}$$

is finite.

The first summation is finite as it consists of a finite number of terms. In the second summation note that each term is less than $|x[n]|r_0^{-n}$ as. Since $\sum_{n=1}^{\infty} |x[n]|r_0^{-n}$ is finite by our assumption that circle with radius r_0 lies in ROC, the second sum is also finite. Hence values of z such that $|z| > r_0$ lies in ROC, except when. At $z = \infty$, the first summation will became infinite. So if $n_0 \geq 0$, i.e. the sequence $x[n]$ is causal, the value $z = \infty$ will lie in the ROC.

5. If $\{x[n]\}$ is left sided sequence, i.e. $x[n] = 0, n > n_0$ and $z| = r_0$ lies in the ROC, the values of z function $0 < |z| < r_0$ also lie in the ROC. The proof is similar to the property 4. The point $z = 0$, will lie in the ROC if the sequence is purely anticausal $(x[n] = 0, n > 0)$
6. If $\{x[n]\}$ is non zero for, $n_1 \leq n \leq n_2$, then ROC is entire z-plane except possibly $z = 0$, and/or. In this case the $X(z)$ consists of finite number of terms and therefore it converges if each term infinite which is the case when z is different from 0 or. $z = 0$ lies in ROC, if $n_2 \leq 0$, and $z = \infty$ lies in the ROC if.
7. If $\{x[n]\}$ is two-sided sequence and if circle $z| = r_0$ is in ROC, then ROC will consist of annular region in z-plane, which includes. We can express a two sided sequence as sum of a right sided sequence and a left sided sequence. Then using property 4 and 5 we get this property. Using property 2 and 3 we see what ROC will be banded by circles passing through the poles.

The inverse z-transform

The inverse z-transform is given by

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

the symbol $\oint$ indicates contour integration, over a counter clockwise contour in the ROC of. If $X(z)$ ratio of polynomials one can use Cauchy integral theorem to calculate the contour integral. There are alternative procedures also, which will be considered after discussing the properties of z-transform.

Properties of the z-transform

We use the notation

$$\{x[n]\} \leftrightarrow X(z), \qquad ROC = R_x$$

to denote z-transform of the sequence.

1. Linearity

The z-transform of a linear combination of two sequence is given by

$$a\{x[n]\} + b\{y[n]\} \leftrightarrow aX(z) + bY(z), \qquad \text{ROC contains } (R_x \cap Ry)$$

The algebraic form follows directly from the definition, equation (7.2). The linear combination is such that some zero's can cancel the poles, then the region of convergence may be larger. For example if the linear combination $\{a^n u[n] - a^n u[n-N]\}$ is a finite-length sequence, the ROC is entire z-plane except at $a = 0$, like individual ROCs are. If the intersection of R_x and R_y is null set, the z-transform of the linear combination will not exist.

2. Time shifting

If we shift the time sequence, we get $\{x[n-n_0]\} \leftrightarrow z^{-n_0}X(z)$, $ROC = R_x$ except for possible addition or deletion of $z = 0$ and/or $z = \infty$

We have

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}$$

changing variable, $m = n - n_0$

$$Y(z) = \sum_{m=-0}^{\infty} x[m]z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m}$$

$$= z^{-n_0}X(z)$$

The factor z^{-n_0} can affect the poles and zeros at $z = 0$, $z = \infty$

3. Multiplication by a exponential sequence

$$\{z_0^n x[n]\} \leftrightarrow X(z/z_0), \quad ROC = \{z : z/z_0 \in R_x\}$$

This follows directly from defining equation (7.2).

4. Differentiation of $X(z)$:

If we differentiate $X(z)$ term by term we get

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1}$$

Thus

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n\,x[n]z^{-n}$$

$\{nx[n]\} \leftrightarrow -z\frac{dX(z)}{dz},\ ROC = R_x$, except possibly $z = 0,\ z = \infty$

The ROC does not change (except $z = 0$, $z = \infty$). This follows from the property that $X(z)$ is an analytic function.

5. Conjugation of a complex sequence

$$\{x^*[n]\} \leftrightarrow X^*(z^*),\ ROC = R_x$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x^*[n]z^{-n}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]\,(z^*)^{-n}\right)^*$$

$$= X^*(z^*)$$

Since ROC depends only an magnitude $|z|$ it does not change.

6. Time Reversal

$$\{x[-n]\} \leftrightarrow X(1/z)$$

$$ROC = \{z : \frac{1}{z}\ \epsilon R_X\}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

We have

putting $m = -n$

$$y(z) = \sum_{m=-\infty}^{\infty} x[m] z^{m}$$

$$= X(1/z)$$

If we combine it with the previous property, we get

$$\{x^*[-n]\} \leftrightarrow X^*(1/z^*),\ ROC = \{z : \frac{1}{z} \epsilon R_x\}$$

7. Convolution of sequence

$$\{x[n]\} * \{y[n]\} \leftrightarrow X(z)Y(z),$$ ROC contains $R_x \cap R_y$

The z-transform of the convolution is

$$\sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k] y[n-k] \right) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k] z^{-n}$$

Interchanging the order of summation
using time shifting property (or changing index of summation)

$$= \sum_{k=-\infty}^{\infty} x[k] z^{-k} Y(z)$$

$$= X(z)Y(z)$$

If there is pole-zero cancelation, the ROC will be larger than the common ROC of two sequence. Convolution property plays an important role in analysis of LTI system. An LTI system, which produces a delay of n_0, has the transfer function z^{-n_0}, therefore delay of n_o units is often depicted by z^{-n_0}

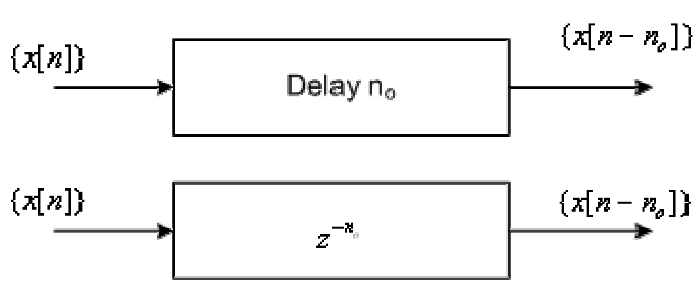


Fig 7.3

8. Complex convolution theorem

If we multiply two sequences then

$$\{x[n]y[n]\} \leftrightarrow \frac{1}{2\pi j}\oint X(v)Y(z/v)v^{-1}dv,$$ ROC contains $\{zw, z\epsilon R_x, w\epsilon R_y\}$

This can be proved using inverse z-transform definition.

9. Initial value Theorem

If $x[n]$ is zero for $n<0$, i.e. $\{x[n]\}$ is causal, then

$$x[0] = \lim_{z\to\infty} X(z)$$

Taking limit term by term in $X(z)$, we get the above result.

10. Parseval's relation

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] \leftrightarrow \frac{1}{2\pi j}\oint X(v)Y^*(1/v^*)v^{-1}dv$$

These properties are summarized in table 7.1

Sequence	Transform	ROC
1. $\{x[n]\}$	$X(z)$	R_x
2. $a\{x[n]\}+b\{y[n]\}$	$aX(z)+bY(z)$	contains $R_x \cap R_y$
3. $\{x[n-n_0]\}$	$z^{-n_o}X(z)$	R_x, except change at $z=o, z=\infty$
4. $\{z_0^n x[n]\}$	$X(z/z_0)$	$\{z/z_0 \epsilon R_x\}$
5. $\{nx[n]\}$	$-z\frac{dX(z)}{dz}$	R_x, except change at $-z=0, z=\infty$
6. $\{x^*[n]\}$	$X^*(z^*)$	R_x
7. $\{x[-n]\}$	$X(1/z)$	$\{1/Z\epsilon R_x\}$
8. $\{x[n]\}*\{y[n]\}$	$X(z)Y(z)$	Contains $R_x \cap R_y$
9. $\{x[n]y[n]\}$	$\frac{1}{2\pi j}\oint X(v)Y(z/v)u_{dv}^{-1}$	Contains $R_x R_y$

Table 7.1 z-transform properties

Methods of inverse z-transform

We can use the contour integration and the equation (7.6) to calculate inverse z-transform. This equation has to be evaluated for all values of n, which can be quite complicated in many cases. Here we give two simple methods for the inverse transform computation.

1. Inverse transform by partial fraction expansion

This is method is useful when z-transform is ratio of polynomials. A rational $X(z)$ can be expressed as

$$X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in. If degree M of the numerator polynomial $N(z)$ is greater than or equal to the degree N of the denominator polynomial $D(z)$, we can divide $N(z)$ by $D(z)$ and re-express $X(z)$ as

$$X(z) = \sum_{k=0}^{M-N} a[k]z^{-k} + \frac{N_1(z)}{D(z)}$$

where the degree of polynomial $N_1(z)$ is strictly less than that of. For simplicity let us assume that

$$X(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k Z^{-l}}$$

all poles are simple. Then

$$A_k = (1 - d_k Z^{-1}) \frac{N_1(z)}{D(z)} |_{Z=d_k}$$

where

Example: Let

$$X(z) = \frac{1 + 2z^{-1}}{(1 - \cdot 2\, z^{-1})(1 + \cdot 6z^{-1})}$$

The partial fraction expression is

$$X(z) = \frac{A_1}{1 - \cdot 2\, z^{-1}} + \frac{A_2}{(1 + \cdot 6\, z^{-1})}$$

$$A_1 = (1 - \cdot 2\, z^{-1})X(z)|_{Z=\cdot 2} = \frac{1 + 2z^{-1}}{1 + \cdot 6\, z^{-1}}|_{z=\cdot 2} = 2.75$$

$$A_2 = (1 - \cdot 6\, z^{-1})X(z)|_{Z=-\cdot 6} = \frac{1 + 2z^{-1}}{1 - \cdot 2z^{-1}}|_{z=\cdot -6} = -1.75$$

$$X(z) = \frac{2.75}{1 - \cdot 2z^{-1}} - \frac{1.75}{1 + \cdot 6z^{-1}}$$

The inverse z-transform depends on the ROC. If ROC is $|z| > \cdot 6$, then ROCs associated with each term is outside a circle(so that common ROC is outside a circle), sequences are causal. Using linearity property and z-transform of $a^n u[n]$ we get

$$x[n] = 2.75(0.2)^n u[n] - 1.75(-.6)^n u[n]$$

If the ROC is $.2 < |z| < .6$, the ROC of the term $\frac{1}{1-.2z^{-1}}$ should be outside the circle $z| = .2$, and ROC for $\frac{1}{1+.6z^{-1}}$ should be. Hence we get the sequence as

$$x[n] = 2.75(.2)^n u[n] + 1.75(-.6)^n u[-n-1]$$

Similarly if ROC is $|Z| < .2$ we get a noncausal sequence

$$x[n] = -2.75(.2)^n u[-n-1] + 1.75(-.6)^n u[-n-1]$$

If $X(z)$ has multiple poles, the partial fraction has slightly different form. If $X(z)$ has a pole of order s at $z = d_i$, and all other poles are simple Then

$$X(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1-d_k Z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}$$

where A_k and B_k are obtained as before, the coefficients C_m are given by

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} (1-d_i w)^s X(w^{-1}) \right\}_{w=d_i^{-1}}$$

If there are more multiple poles, there will be more terms like the third term. Using linearity and differentiation properties we get some useful z-transform pairs given in Table 7.2

Sequence	Transform	ROC
1. $\{\delta[n]\}$	1	All z
2. $\{\delta[n-m]\}$	z^{-m}	All z, except 0(if $m > 0$) or ∞(if $m < 0$)
3. $\{a^n u[n]\}$	$\frac{1}{1-az-1}$	$\lvert z\rvert > \lvert a\rvert$

4. $\{-a^n u[-n-1]\}$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
5. $\{na^n u[n]\}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6. $\{-na^n u[-n-1]\}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7. $\{r^n \cos w_0 n\, u[n]\}$	$\frac{1-r\cos w_0 z^{-1}}{1-2r\cos w_0 z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$
8. $\{r^n \sin w_0 n\, u[n]\}$	$\frac{\sin w_0 z^{-1}}{1-2r\cos w_0 z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$
9. $\{a^n, 0 \le n \le N-1\}$	$\frac{1-a^N z^{-N}}{1-a z^{-1}}$	$\lvert z\rvert > 0$

Table 7.2 Some useful z-transform pairs

2. Inverse Transform via long division

For causal sequence the z-transform $X(z)$ can be exported into a pure series in. In the series expansion, the coefficient multiplying the term z^{-n} is. If $X(z)$ is anticausal then we expand in terms of poles of.

Example 1: Let

$$X(z) = \frac{1+2z^{-1}}{(1-.2z^{-1})(1+.6z^{-1})},\ \ ROC\ |z| > .6$$

This is a causal sequence, long division gives

$$
\begin{array}{r|l}
 & 1+1.6z^{-1} - .52z^{-2} + .4z^{-3} + \ldots \\ \hline
1+.4z^{-1} - .12z^{-2} & 1+2z^{-1} \\
 & 1+.4z^{-1} - .12z^{-2} \\ \cline{2-2}
 & 1.6z^{-1} - .12z^{-2} \\
 & 1.6z^{-1} - .64z^{-2} - .192z^{-3} \\ \cline{2-2}
 & -.52z^{-2} - .192z^{-3} \\
 & -.52z^{-2} - .208z^{-3} + .0624z^{-4} \\ \cline{2-2}
 & -.4z^{-3} - .0624z^{-4} \\
 & \ldots\ldots\ldots \\
 & \ldots\ldots\ldots
\end{array}
$$

This gives $x[0] = 1, x[1] = 1.6, x[2] = -.52, x[3] = .4$,.....
We can see that it is not easy to write the n^{th} term.

Example 2:

$$X(z) = ln(1 + az^{-1}),\ |z| > |a|$$

Using the pure series expansion for $ln(1+x)$ with $|x| < |$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n}, & n \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Analysis of LTI system using z-transform

From the convolution property we have

$$Y(z) = H(z)\,X(z)$$

where $X(z), Y(z)$ are $H(z)$ are z-transforms of input sequence $\{x[n]\}$, output sequence $\{y[n]\}$ and impulse response $\{h[n]\}$ respectively. The $H(z)$ is referred to as system function or transfer function of the system. For z on the unit circle $(z = e^{jw})$, $H(z)$ reduces to the frequency response of the system, provided that unit circle is in the ROC for.

A causal LTI system has impulse response $\{h[n]\}$ such that. Thus ROC of $H(z)$ is exterior of a circle in z-plane including. Thus a discrete time LTI system is causal if and only if ROC is exterior of a circle which includes infinity.

An LTI system is stable if and only if impulse response $\{h[n]\}$ is absolutely summable. This is equivalent to saying that unit circle is in the ROC of.

For a causal and stable system ROC is outside a circle and ROC contains the unit circle. That means all the poles are inside the unit circle. Thus a causal LTI system is stable if and if only if all the poles inside unit circle.

LTI systems characterized by Linear constant coefficient difference equation

For the system characterized by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

We take the z-transform of both sides and use linearity and the time shift property to get

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Thus the system function is always a rational function. We can write it by inspection. Numerator polynomial coefficients are the coefficients of $x[n-k]$ and denominator coefficients are coefficients of. The difference equation by itself does not provide information about the ROC, it can be determined by conditions like causality and stability.

System Function and block diagram representation

The use of z-transform allows us to replace time domain operation such as convolution time shifting etc with algebraic operations.
Consider the parallel interconnection if two system, as shown in figure 7.4.

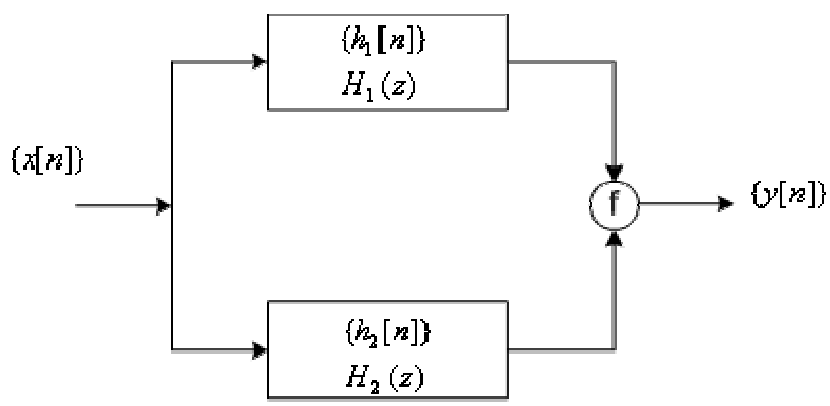


Fig 7.4

The impulse response of the over all system is

$$\{h[n]\} = \{h_1[n]\} + \{h_2[n]\}$$

From linearity of the z-transform,

$$H(z) = H_1(z) + H_2(z)$$

Similarly, the impulse response of the series connection in figure 7.5 is

$$\{h[n]\} = \{h_1[n]\} * \{h_2[n]\}$$

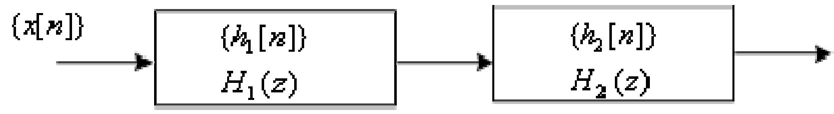


Fig 7.5

From the convolution property.

$$H(z) = H_1(z)H_2(z)$$

The z-transform of the interconnection of linear system can be obtained by algebraic means. For example consider the feed back connection in figure 7.6

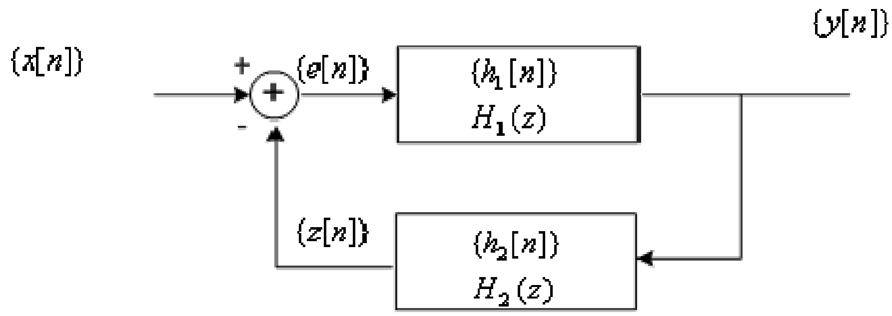


Fig 7.6

We have

$$Y(z) = H_1(z)E(z)$$

$$E(z) = X(z) - Z(z)$$

$$= X(z) - Y(z)H_2(z)$$

$$Y(z) = H_1(z)[X(z) - Y(z)H_2(z)]$$

or

$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

Even though this course is primarily about the discrete time signal processing, most signals we encounter in daily life are continuous in time such as speech, music and images. Increasingly discrete-time signals processing algorithms are being used to process such signals. For processing by digital systems, the discrete time signals are represented in digital form with each discrete time sample as binary word. Therefore we need the analog to digital and digital to analog interface circuits to convert the continuous time signals into discrete time digital form and vice versa. As a result it is necessary to develop the relations between continuous time and discrete time representations.

Sampling of continuous time signals

Let $\{x_c(t)\}$ be a continuous time signal that is sampled uniformly at t = nT generating the sequence $\{x[n]\}$ where

$$x[n] = x_c(nT), \qquad -\infty < n < \infty, \quad T > 0$$

T is called sampling period, the reciprocal of T is called the sampling frequency. The frequency domain representation of $\{x_c(t)\}$ is given by its Fourier transform

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt$$

where the frequency-domain representation of $\{x[n]\}$ is given by its discrete time fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

To establish relationship between the two representation, we use impulse train sampling. This should be understood as mathematically convenient method for understanding sampling. Actual circuits can not produce continuous time impulses. A periodic impulse train is given by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{8.1}$$

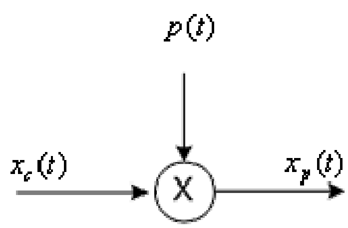
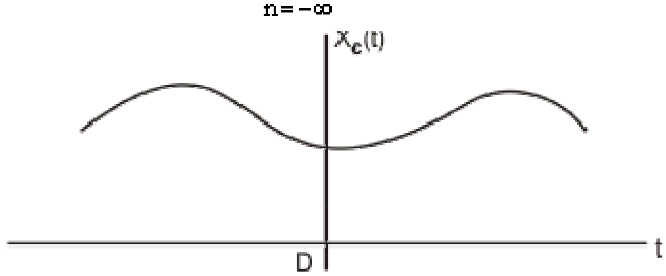


Fig 8.1

$$x_p(t) = x_c(t)p(t) \tag{8.2}$$

using sampling property of the impulse $f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$, we get

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t - nT) \tag{8.3}$$



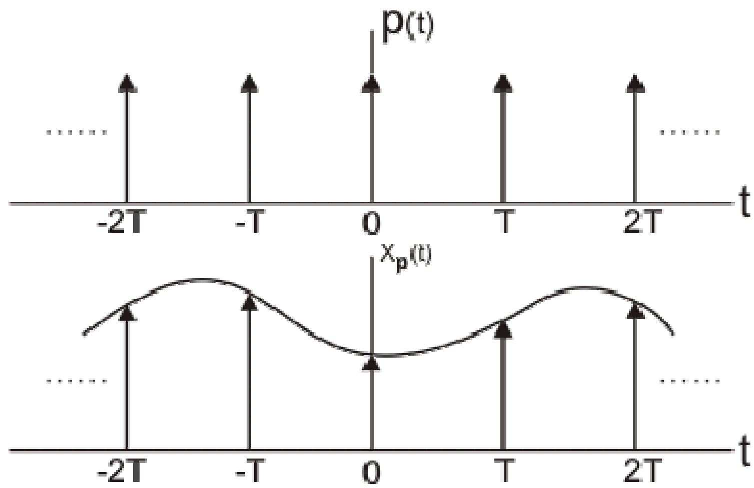


Fig 8.2

From multiplication property, we know that

$$x_p(j\Omega) = \frac{1}{2\pi}[X_c(j\Omega) * P(j\Omega)]$$

The Fourier transform of a impulse train is given by

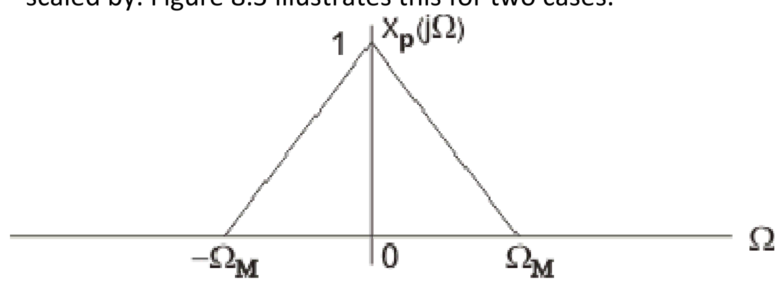
$$P(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$$

where $\Omega_s = \frac{2\pi}{T}$

Using the property that $X(j\Omega) * \delta(\Omega - \Omega_0) = X(j(\Omega - \Omega_0))$ it follows that

$$X_p(j\Omega) = \frac{1}{\mathbf{T}}\sum_{k=-\infty}^{\infty}X_c(j\Omega - k\Omega_s) \tag{8.4}$$

Thus $X_p(j\Omega)$ is a periodic function of Ω with period Ω_s, consisting of superposition of shifted replicas of $X_c(j\Omega)$ scaled by. Figure 8.3 illustrates this for two cases.



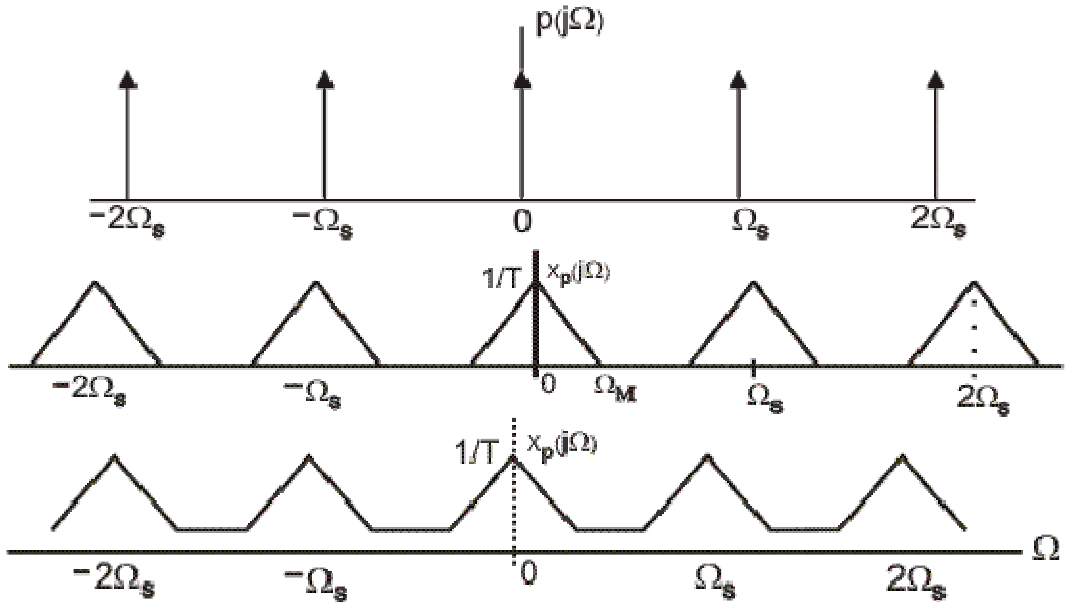


Fig 8.3

If $\Omega_m < (\Omega_s - \Omega_m)$ or equivalently $\Omega_s > 2\Omega_m$ there is no overlap between shifted replicas of $X_c(j\Omega)$, where as with $\Omega_s > 2\Omega_m$, there is overlap. Thus if $\Omega_s > 2\Omega_m$, $X_c(j\Omega)$ is faithfully replicated in $X_p(j\Omega)$ and can be recovered from $x_p(t)$ by means of lowpass filtering with gain T and cut off frequency between Ω_m and. This result is known as Nyquist sampling theorem.

Sampling Theorem

Let $x_c(t)$ be a bandlimited signal with $X_c(j\Omega) = 0$, for. Then $X_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT), -\infty < n < \infty$, if

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_m$$

The frequency $2\Omega_m$ is called Nyquist rate, while the frequency Ω_m is called the Nyquist frequency. The signal $x_c(t)$ can be reconstructed by passing $x_p(t)$ through a lowpass filter.

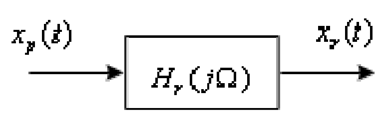


Fig 8.4

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| \ge \Omega_c \end{cases}$$

The impulse response of this filter is

$$h_r(t) = \frac{\sin \Omega_c t}{\pi t/T}$$

$$x_r(t) = x_p(t) * h_r(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT) \right) * h_r(t)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\,h_r(t-nT) \qquad (8.5)$$

Assuming $\Omega_c = \Omega_s/2 = \pi/T$ we get

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\frac{\sin \pi(t-nT)/T}{\pi(t-nT)/T} \qquad (8.6)$$

The above expression (8.5) shows that reconstructed continuous time signal $x_r(t)$ is obtained by shifting in time the impulse response of low pass filter $h_r(t)$ by an amount *nT* and scaling it in amplitude by a factor $x[n] = x_c(nT)$ for all integer values n. The interpolation using the impulse response of an ideal low pass filter in (8.6) is referred to as bandlimited interpolation, since it implements reconstruction if $x_c(t)$ is bandlimited and sampling frequency satisfies the condition of the sampling theorem. The reconstruction is in the mean square sense i.e.

$$\int_{-\infty}^{\infty} (x_c(t) - x_r(t))^2\,dt = 0$$

The effect of underselling: Aliasing

We have seen earlier that spectrum $X_c(j\Omega)$ is not faithfully copied when. The terms in (8.4) overlap. The signal $x_c(t)$ is no longer recoverable from. This effect, in which individual terms in equation (8.4) overlap is called aliasing.
For the ideal low pass signal

$$h_r(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Hence $x_r(nT) = x_c(nT), \quad n = 0, \pm 1, \pm 2 \ldots\ldots$
Thus at the sampling instants the signal values of the original and reconstructed signal are same for any sampling frequency.

DTFT of the discrete time signal

Taking continuous time Fourier transform of equation (8.3) we get

$$X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega nT} \qquad (8.7)$$

Since $x[n] = x_c(nT)$, we get the DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8.8}$$

comparing them we see that

$$X_p(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T})$$

using equation (8.4) we get

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

or

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \tag{8.9}$$

Comparing equation (8.4) and(8.9) we see that $X(e^{j\omega})$ is simply a frequency scaled version of $X_p(j\Omega)$ with frequency scaling specified by. This can be thought of as a normalization of frequency axis so that frequency $\Omega = \Omega_s$ in $X_p(j\Omega)$ is normalized to $\omega = 2\pi$ in. For the example in figure 8.3 the $X(e^{j\omega})$ is shown in figure (8.5)
From equation (8.5) we see that

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] h_r(j\Omega) e^{-j\Omega T n}$$

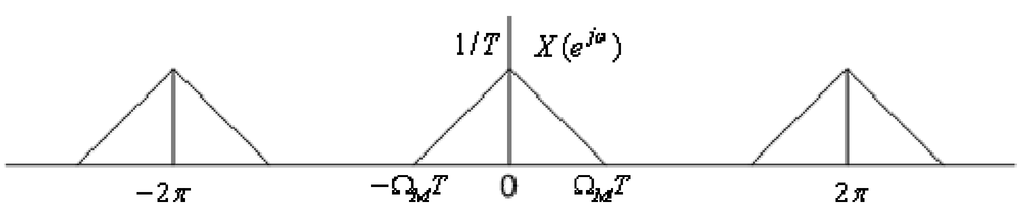


Fig 8.5

$$= H_r(j\Omega) X(e^{j\Omega T}) \tag{8.10}$$

We refer to the system that implements $x[n] = x_c(nT)$ as ideal continuous-to-discrete time (C/D) convertor and is depicted in figure (8.6)

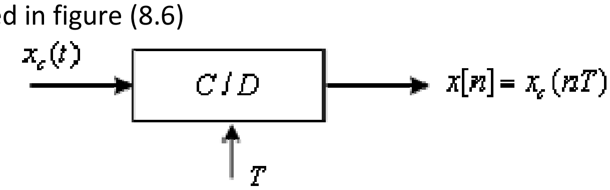


Fig 8.6

The ideal system that takes $\{x[n]\}$ sequence as input and produces $x_r(t)$ given equation (8.5) is called ideal discrete to continuous time (D/C) convertor and is depicted in Figure (8.7)

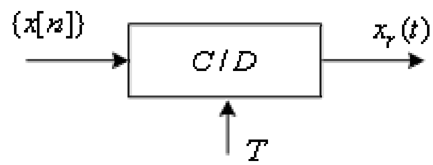


Fig 8.7

Discrete time processing of continuous time signal

Figure (8.8) shows a system for discrete time processing of continuous time system

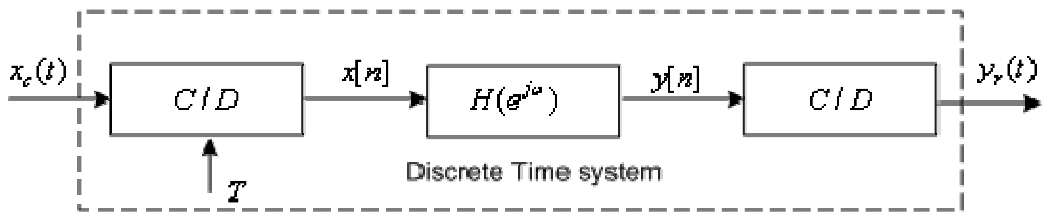


Fig 8.8

The over all system has $x_c(t)$ as input and $y_r(t)$ as output. We have the following relations among the signals.

$$x[n] = x_c(nT)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\sin\frac{\pi(t-nT)/T}{\pi(t-nT)/T}$$

and

$$Y_r(j\Omega) = H_r(j\Omega)\,Y(e^{j\Omega T})$$

$$= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

If the discrete time system is LTI then we have

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

combining these equations we get

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

$$= H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

(8.11)

If $X_c(j\Omega) = 0$, for $|\Omega| \geq \pi/T$ and we use ideal lowpass reconstruction filter then only the term for k = 0 is passed by the filter and we get

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Thus if $X_c(j\Omega)$ is bandlimited and sampling rate is above the Nyquist rate, the output is related to the input by

$$Y_r(j\Omega) = H_{eq}(j\Omega)X_{cv}(j\Omega)$$

where

$$H_{eq}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

(8.12)

That is overall system is equivalent to a linear time invariant system for bandlimited signal.

The LTI property of the system depends on two factors. First the discrete time system is LTI and second the input signals are bandlimited to half the sampling frequency

Example

Let us consider the system in figure 8.8 with

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

The frequency response is periodic with period. For a bandlimited input signal, sampled above the Nyquist rate, the overall system will behave like a LTI continuous time system with

$$H_{eq}(j\Omega) = \begin{cases} 1 & |\Omega T| < \omega_c \;\text{ or }\; |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$

Thus the equivalent system is ideal lowpass system with cut off frequency. With a fixed discrete time filter by changing T we can change the cut off frequency of the equivalent system. Spectra for various signals are depicted in figure 8.9.

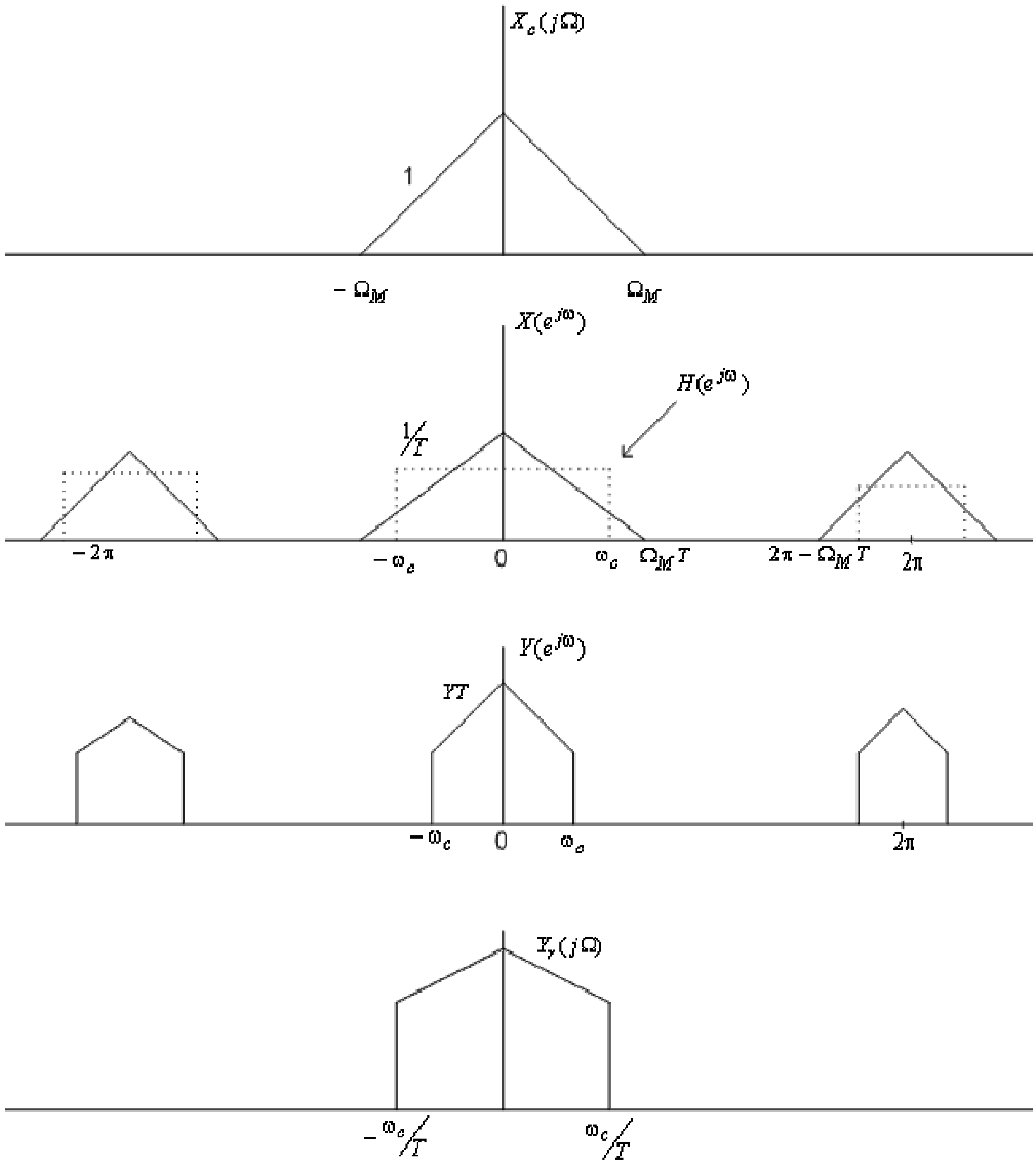


FIGURE 8.9

From figure (8.9) we can see that ever if there is same aliasing due to sampling, if the components are filtered out by the discrete time system, the over all transfer function will remain same. Thus the requirement is

$(2\pi - \Omega_m T) > \omega_c$ instead of $(2\pi - \Omega_m T) > \Omega_m T$ for no aliasing.

Continuous time processing of discrete time signals

Consider the system shown in figure (8.10)

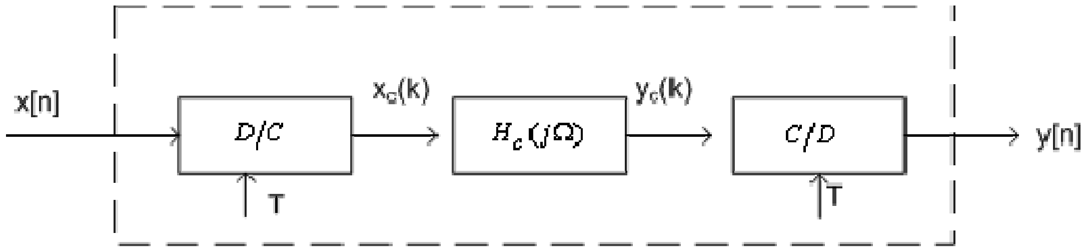


Figure 8.10

We have

$$X_c(j\Omega) = TX(e^{j\Omega T}), \qquad |\Omega| < \pi/T$$

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega), \qquad |\Omega| < \pi/T$$

$$Y(e^{j\omega}) = \frac{1}{T}Y_c(j\frac{\omega}{T}), \qquad |\omega| < \pi$$

$$= H_c(j\frac{\omega}{T})X(e^{j\omega}),$$

Therefore the overall system behaves as a discrete time system where frequency response is

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \qquad |\omega| < \pi \tag{8.13}$$

Example

Let us consider a discrete time system with frequency response

$$H(e^{j\omega}) = e^{-j\omega\Delta}, \qquad |\omega| < \pi$$

when Δ is an integer, this system is delay by Δ

$$y[n] = x[n - \Delta]$$

but when Δ is not an integer, we can not write the above equation. Suppose that we implement this using system in figure (8.10). Then we have

$$H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega\Delta T} \tag{8.14}$$

So that overall system has frequency response. The equation (8.13) represents a time delay ΔT secs in continuous time whether Δ is integer or not, thus

$$Y_c(t) = x_c(t - \Delta T)$$

The signal $x_c(t)$ is bandlimited interpolation of $x[n]$ and $y[n]$ is obtained by sampling. Thus *y*[*n*] are samples of band limited signal $x_c(t)$ delayed by.

$$y[n] = y_c(nT) = x_c(nT - \Delta T)$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot \frac{\sin[\pi(t - \Delta T - kT/T]}{[\pi(t - \Delta T - kT/T]}\Big|_{t=nT}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n-k-\Delta)]}{\pi(n-k-\Delta)}$$

For $\Delta = 1/2,\ \{y[n]\}$ are depicted in figure (8.11)

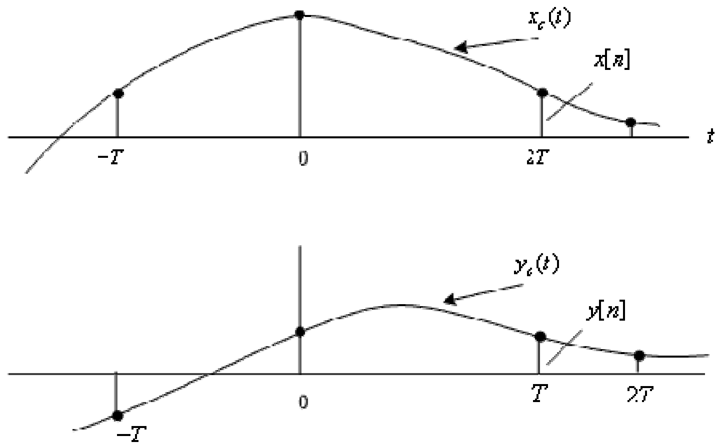
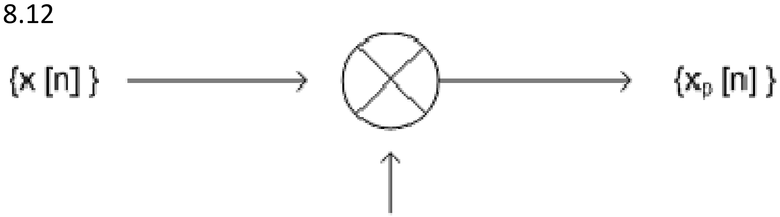


Fig 8.11

Sampling of discrete time Signals

In analogy with continuous time sampling, the sampling of a discrete time signal can be represented as shown in figure 8.12



$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

FIGURE 8.12

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

$$x_p[n] = \begin{cases} x[n], & \text{if n is integer multiple of N} \\ 0, & \text{otherwise} \end{cases}$$

(8.15)

$$x_p[n] = x[n]\,p[n]$$

$$= \sum_{k=-\infty}^{\infty} x[kN]\delta[n-kN]$$

In frequency domain, we get

$$X_p(e^{j\omega}) = \frac{1}{2}\int_{-\pi}^{\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

The Fourier transform of $\{p[n]\}$ sequence is

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s),$$

where

$$\omega_s = \frac{2\pi}{N}$$

Thus we get

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)}) \tag{8.16}$$

Figure 8.13 illustrates signals and their spectra

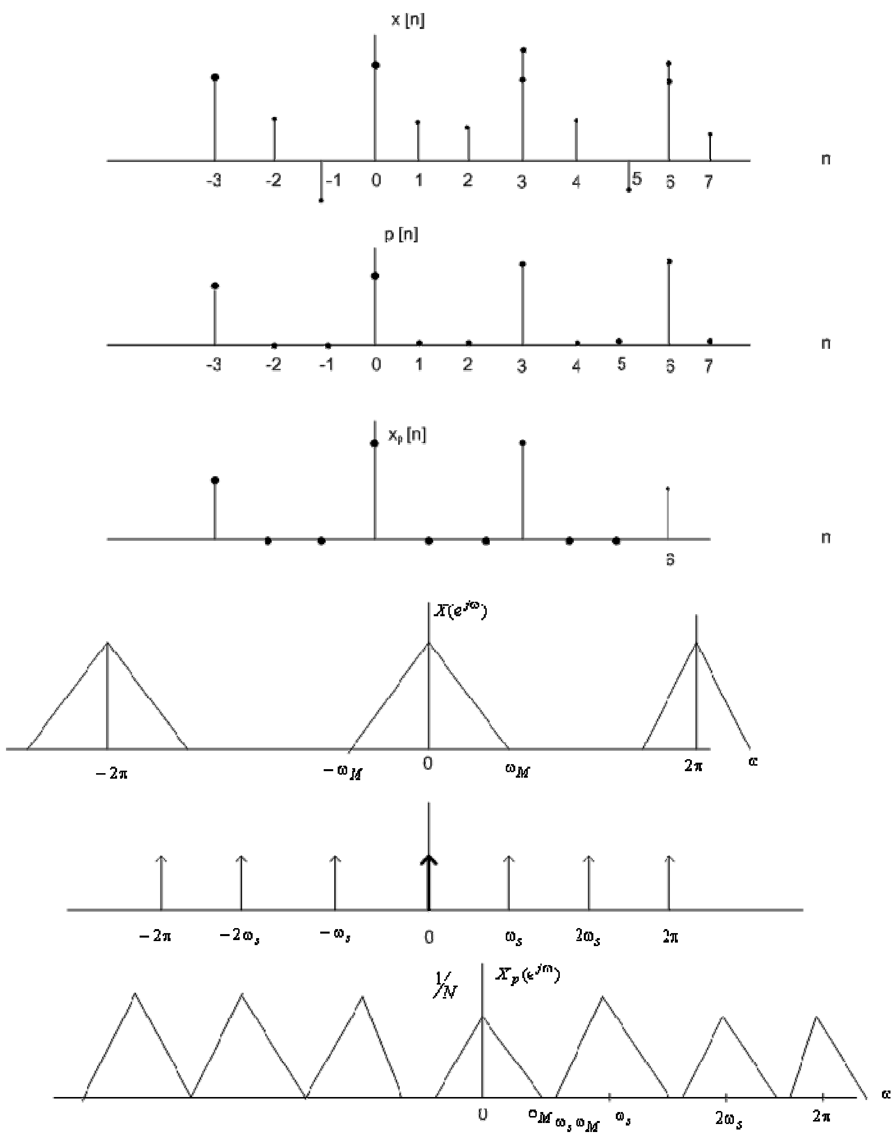


FIGURE 8.13

If $(\omega_s - \omega_m) > \omega_m$ or equivalently $\omega_s > 2\omega_m$ or $\frac{2\pi}{N} > 2\omega_m$ there will be no aliasing (i.e non zero portions of $X(e^{j\omega})$ do not overlap) and the signal $\{x[n]\}$ can be recovered from $x_p[n]$ by passing through an ideal low-pass filter with gain equal to *N* and cut off equal to $\omega_s/2$

$$H_r(e^{j\omega}) = \begin{cases} N & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

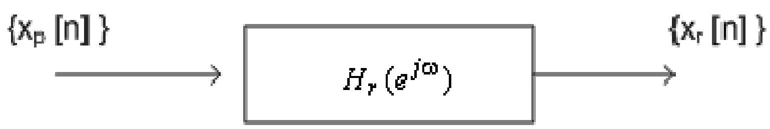


FIGURE 8.14

If $\omega_s < 2\omega_m$, there will be aliasing, and so $\{x_r[n]\}$ will be different from. However as in continuous time case

$x_r[kN] = x[kN],\ k = 0, \pm 1, \pm 2, \ldots$

independently of whether there is aliasing or not.

$$\begin{aligned} x_r[n] &= \{x_p[n]\} * \{h_r[n]\} \\ &= \left\{ \sum_{k=-\infty}^{\infty} x[hN]\delta[n - kN] \right\} * \{h_r[n]\} \\ &= \sum_{k=-\infty}^{\infty} x[kN]h_r[n - kN] \end{aligned}$$

For ideal low pass filter

$$h_r[n] = \frac{N\omega_c}{\pi} \frac{\text{sun } \omega_c n}{\omega_c n}$$

with $\omega_c = \pi/N$ we get

$$x_r[n] = \sum_{k=-\infty}^{\infty} x[kN] \frac{\sin \pi(n - kN)/N}{\pi(n - kN)/N}$$

Discrete time decimation and interpolation

The sampled signal in equation (8.13) has (*N* - 1) samples out of every *N* samples as zeros. We define a new sequence which retains only the non zero values

$$\begin{aligned} x_d[n] &= x_p[nN] \\ &= x[nN] \end{aligned} \quad (8.17)$$

this is called a decimated sequence, whatever may be the value of. The DTFT of the decimated request is given by

$$X_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_p[nN]e^{-j\omega n}$$

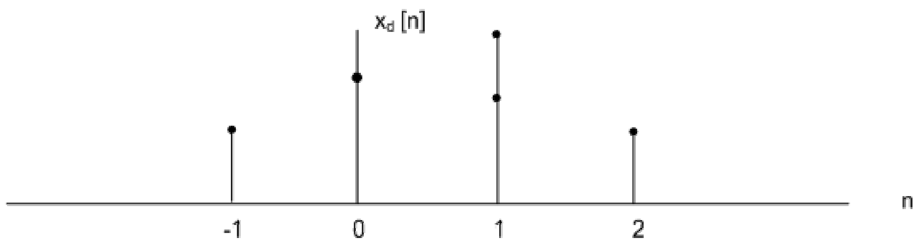
since only for multiples of $N, x_p[n]$ has non zero value,

$$= \sum_{m=-\infty}^{\infty} x_p[m]e^{-j\frac{\omega}{N}m}$$

$$= X_p(e^{j\frac{\omega}{N}})$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\frac{\omega}{N}-\frac{2\pi k}{N})}\right) \qquad (8.18)$$

For the signal shown in figure (8.13) the sequence $\{x_d[n]\}$ and its spectrum are illustrated in figure (8.15)



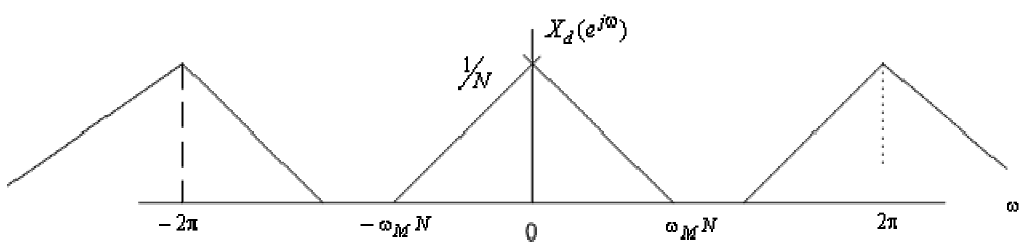


Fig 8.15

If the original signal $\{x[n]\}$ was obtained by sampling a continuous time signal, the process of decimation can be viewed as reduction in the sampling rate by a factor of N. With this interpretation, the process of decimationis often referred as down sampling. The block diagram for this is shown in figure (8.16)

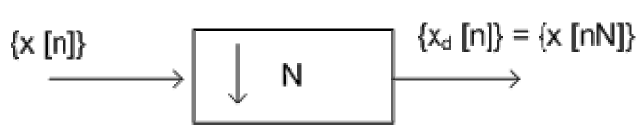


Fig 8.16

There are situations in which it is useful to convert a sequence to a higher equivalent sampling rate. This process is referred to as upsampling or interpolation. This process is reverse of the downsampling. Given a sequence $\{x[n]\}$ we obtain an expanded sequence $x_c[n]$ by inserting (*L* - 1) zero.

$$x_e[n] = \begin{cases} x\left[\dfrac{n}{L}\right], & \text{n multiple of L} \\ 0, & \text{otherwise} \end{cases} \tag{8.19}$$

The interpolated sequence $\{x_i[n]\}$ is obtained by low pass filtering of $\{x_e[n]\}$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_e[n]e^{-j\omega h}$$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m l}$$

$$= X(e^{j\omega L})$$

After low pass filtering

$$X_i(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega L}) \tag{8.20}$$

For ideal low-pass filter with cutoff $\dfrac{\pi}{L}$ and gain *L* we get

$$X_i(e^{j\omega}) = \begin{cases} L\,X(e^{j\omega L}), & |\omega| < \pi/L \\ 0, & \pi/L \le |\omega| \le \pi \end{cases} \tag{8.21}$$

Signals and their spectra interpolation are shown in figure (8.17)

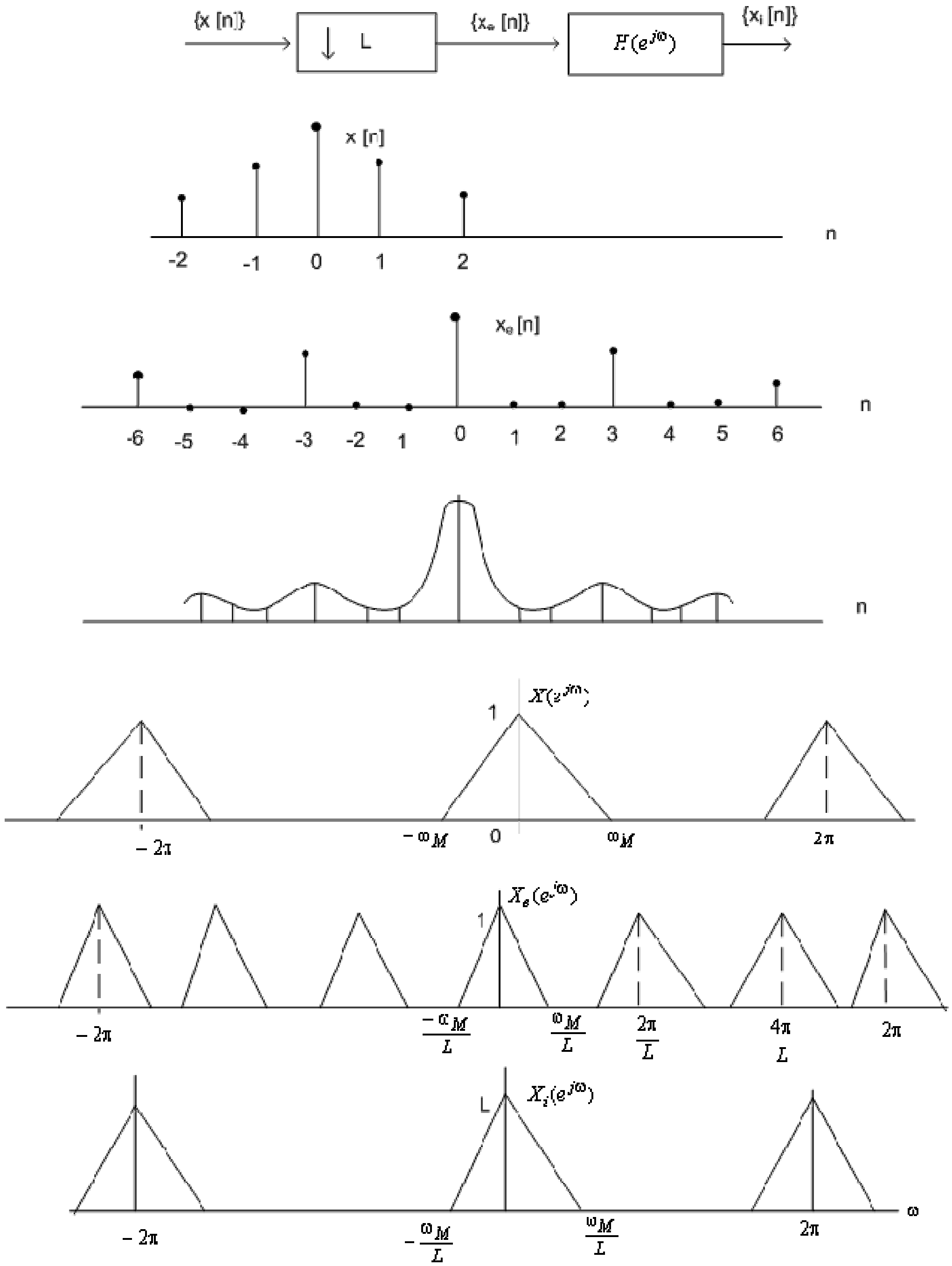


Fig 8.17

We can get a non integer change in rate if it is ratio of two integers by using upsampling and downsampling operations.

In many applications of signal processing we want to change the relative amplitudes and frequency contents of a signal. This process is generally referred to as filtering. Since the Fourier transform of the output is product of input Fourier transform and frequency response of the system, we have to use appropriate frequency response.

Ideal frequency selective filters

An ideal frequency reflective filter passes complex exponential signal. for a given set of frequencies and completely rejects the others. Figure (9.1) shows frequency response for ideal low pass filter (LPF), ideal high pass filter (HPF), ideal bandpass filter (BPF) and ideal backstop filter (BSF).

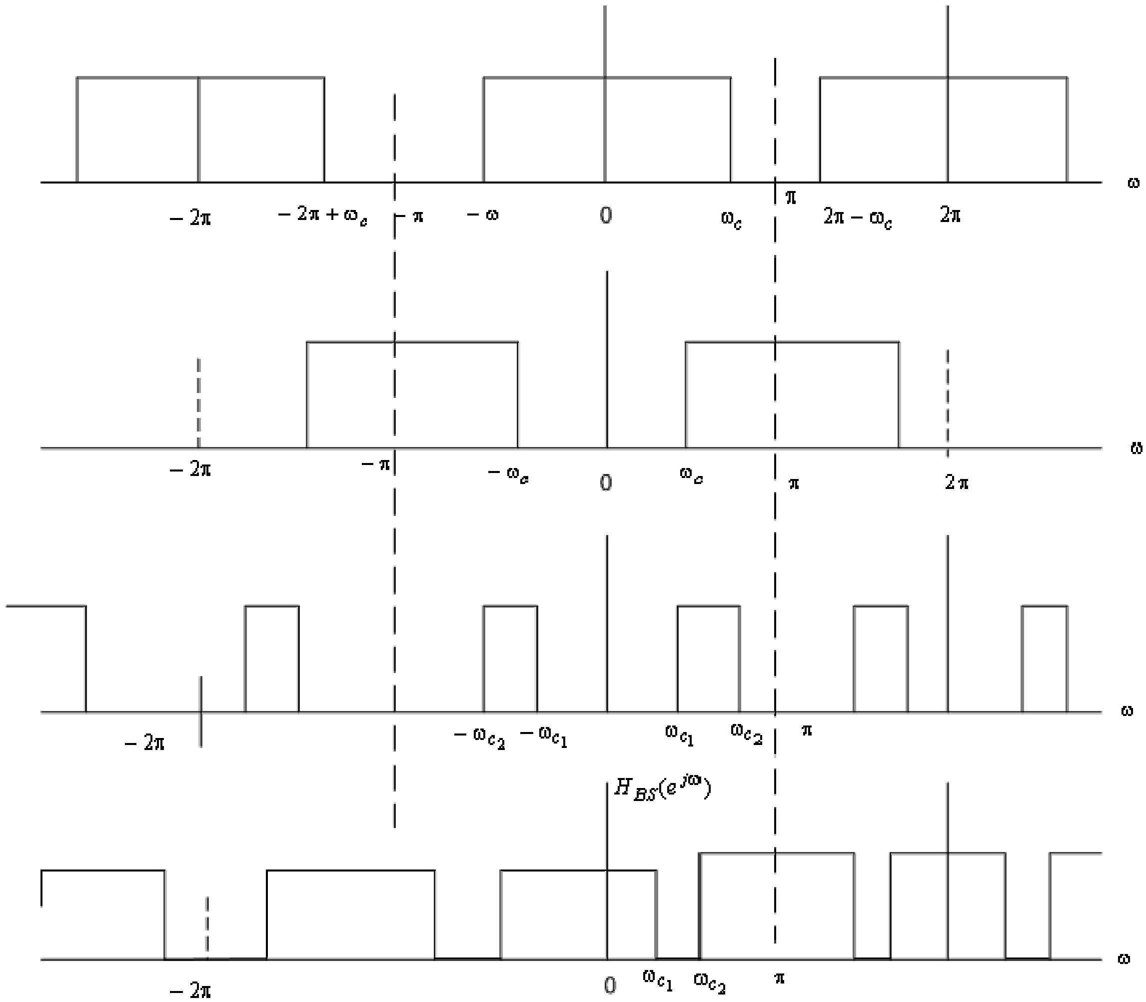


Fig 9.1

The ideal filters have a frequency response that is real and non-negative, in other words, has a zero phase characteristics. A linear phase characteristics introduces a time shift and this causes no distortion in the shape of the signal in the passband.

Since the Fourier transfer of a stable impulse response is continuous function of ω, can not get a stable ideal filter.

Filter specification

Since the frequency response of the realizable filter should be a continuous function, the magnitude re lowpass filter is specified with some acceptable tolerance. Moreover, a transition band is specified between the passband and stop band to permit the magnitude to drop off smoothly. Figure (9.2) illustrates this

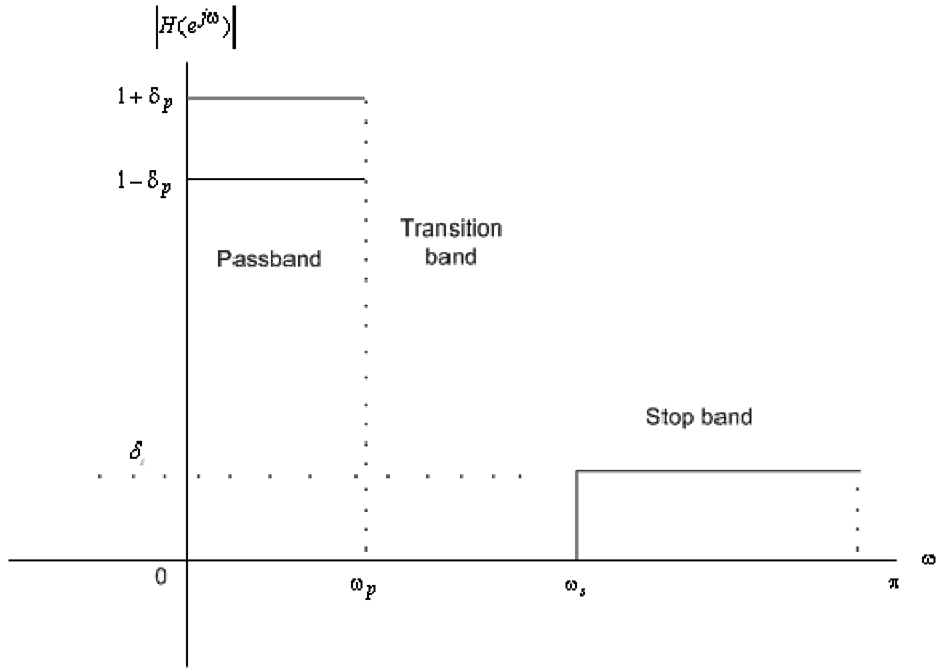


Fig 9.2

In the passband magnitude the frequency response is within $\pm\delta_p$ of unity

$$(1-\delta_p) \le \left|H(e^{j\omega})\right| \le (1+\delta_p), \qquad |\omega| \le \omega_p$$

In the stopband

$$\left|H(e^{j\omega})\right| \le \delta_s, \qquad |\omega| < \omega_s \le \pi$$

The frequencies ω_p and ω_s are respectively, called the passband edge frequency and the stopband edge frequency. The limits on tolerances δ_p and δ_s are called the peak ripple value. Often the specifications of digital filter are given in terms of the loss function $G(\omega) = -20\log_{10}\left|H(e^{j\omega})\right|$, in dB. The loss specification of digital filter are

$$\alpha_p = -20\log_{10}(1-\delta_p)dB$$

$$\alpha_s = -20\log_{10}\delta_s dB$$

Some times the maximum value in the passband is assumed to be unity and the maximum passband deviation, denoted as $\frac{1}{\sqrt{1+E^2}}$ is given the minimum value of the magnitude in passband. The maximum stopband magnitude is denoted by. The quantity α_{max} is given by

$$\alpha_{max} = -20\log_{10}\left(\sqrt{1+E^2}\right)dB$$

These are illustrated in Fig(9.3)

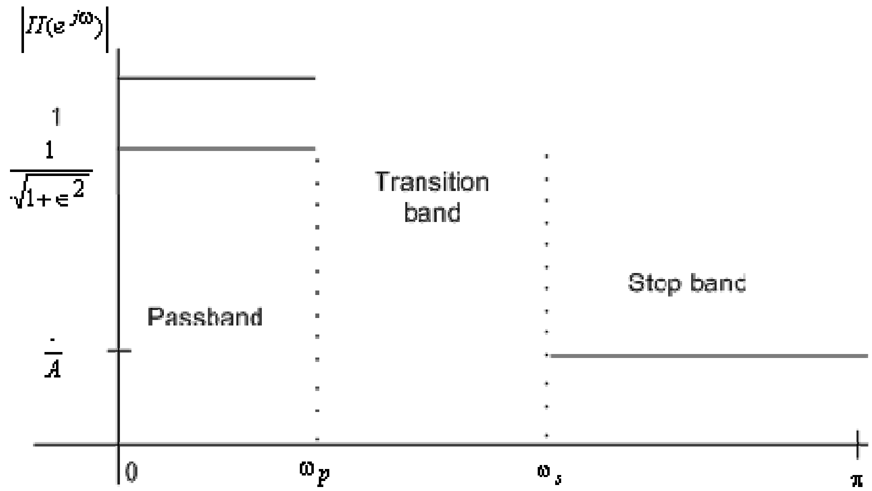


Fig 9.3

If the phase response is not specified, one prefers to use IIR digital filter. In case of an IIR filter design, the most common practice is to convert the digital filter specifications to analog low pass prototype filter specifications, to determine the analog low pass transfer function $H_a(s)$ meeting these specifications, and then to transform it into desired digital filter transfer function. This methods is used for the following reasons:

1. Analog filter approximation techniques are highly advanced.
2. They usually yield closed form solutions.
3. Extensive tables are available for analog-design.
4. Many applications require the digital solutions of analog filters.

The transformations generally have two properties (1) the imaginary axis of the s-plane maps into unit circle of the z-plane and (2) a stable continuous time filter is transformed to a stable discrete time filter.

Filter design by impulse invariance

In the impulse variance design procedure the impulse response of the impulse response $\{h[n]\}$ of the discrete time system is proportional to equally spaced samples of the continues time filter, i.e.,

$$h[n] = T_d h_a(nT_d)$$

where T_d represents a sampling interval, since the specifications of the filter are given in discrete time domain, it turns out that T_d has no role to play in design of the filter. From the sampling theorem we know that the frequency response of the discrete time filter is given by

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a\left(j\frac{\omega}{T_d} + j\frac{2\pi K}{T_d}\right)$$

Since any practical continuous time filter is not strictly bandlimited there issome aliasing. However, if the continuous time filter approaches zero at high frequencies, the aliasing may be negligible. Then the frequency response of the discrete time filter is

$$H(e^{j\omega}) \approx H_a\left(j\frac{\omega}{T_d}\right), \quad |\omega| \leq \pi$$

We first convert digital filter specifications to continuous time filter specifications. Neglecting aliasing, we get $H_a(j\Omega)$ specification by applying the relation

$$\Omega = \omega / T_d \tag{9.2}$$

where $H_a(j\Omega)$ is transferred to the designed filter $H(z)$, we again use equation (9.2) and the parameter $T_d$cancels out.

Let us assume that the poles of the continuous time filter are simple, then

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

The corresponding impulse response is

$$h_a(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Then

$$h[n] = T_d h_a(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n]$$

The system function for this is

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{(1 - e^{s_k T_d} z - 1)}$$

We see that a pole at $s = s_k$ in the s-plane is transformed to a pole at $z = e^{s_k}$ T_d in the z-plane. If the continuous time filter is stable, that is $\mathrm{Re}\{s_k\} < 0$, then the magnitude of $e^{s_k T_d}$ will be less than 1, so the pole will be inside unit circle. Thus the causal discrete time filter is stable. The mapping of zeros is not so straight forward.

Example:

Design a lowpass IIR digital filter $H(z)$ with maximally flat magnitude characteristics. The passband edge frequency ω_p is 0.25π with a passband ripple not exceeding 0.5dB. The minimum stopband attenuation at the stopband edge frequency ω_s of 0.55π is 15 *dB.*

We assume that no aliasing occurs. Taking $T_d = 1$, the analog filter has $\Omega_p = 0.25, \Omega_s = 0.55\pi$, the passband ripple is 0.5dB, and minimum stopped attenuation is 15dB. For maximally flat frequency response we choose Butterworth filter characteristics. From passband ripple of 0.5 dB we get

$$20\log_{10}\left|H_a(j\,0.25\pi)\right| = -0.5dB$$

$$\left|H_a(j\Omega_p)\right|^2 = \frac{1}{1 + \left(\frac{\Omega_p{}^2}{\Omega_c}\right)} = \frac{1}{1 + \epsilon^2}$$

at passband edge.

From this we get $\in^2 = 0.122$

From minimum stopband attenuation of 15 dB we get

$$\left|H_a(j\Omega_p\right|^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^2} = \frac{1}{A^2}$$

at stopped edge $A^2 = 31.62$

The inverse discrimination ratio is given by

$$\frac{1}{k_1} = \frac{\sqrt{A^2 - \mathbf{1}}}{\in} = 15.84$$

and inverse transition ratio $1/k$ is given by

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_s} = 2.2$$

$$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.50$$

Since *N* must be integer we get *N=4.* By $\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \in^2$ we get $\Omega_c = 1.02$

The normalized Butterworth transfer function of order 4 is given by

$$H_{an}(s) = \frac{1}{(s^2 + .7654s + 1)(s^2 + 1.8478s + 1)}$$

$$= \frac{-0.92s - 0.707}{s^2 + .7654s + 1} + \frac{0.92s + 1.707}{s^2 + \mathbf{1}.848s + 1}$$

This is for normalized frequency of 1 rad/s. Replace *s* by $\frac{s}{\Omega_c}$ to get $\mathrm{H_a}(s)$, from this we get

$$\mathrm{H}(z) = \frac{-0.94 + 0.16z^{-1}}{1 - .79z^{-1} + .45z^{-2}} + \frac{.94 - .00167z^{-1}}{1 - .71Z^{-1} + 0.15Z^{-2}}$$

Bilinear Transformation

This technique avoids the problem of aliasing by mapping $j\Omega$ axis in the s-plane to one revaluation of the unit circle in the z-plane.

If $\mathrm{H_a}(s)$ is the continues time transfer function the discrete time transfer function is detained by replacing *s* with

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{9.3}$$

Rearranging terms in equation (9.3) we obtain.

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$$

Substituting $s = \sigma + j\Omega$, we get

$$z = \frac{1+\sigma\frac{T_d}{2}+j\frac{\Omega T_d}{2}}{1-\sigma\frac{T_d}{2}-j\frac{\Omega T_d}{2}}$$

If $\sigma < 0$, it is then magnitude of the real part in denominator is more than that of the numerator and so. Similarly if $\sigma > 0$, than $|z| > 1$ for all. Thus poles in the left half of the s-plane will get mapped to the poles inside the unit circle in z-plane. If $\sigma = 0$ then

$$z = \frac{1+j\frac{\Omega T_d}{2}}{1-j\frac{\Omega T_d}{2}}$$

So, $|z| = 1$, writing $z = e^{j\omega}$ we get

$$e^{j\omega} = \frac{1+j\Omega\frac{T_d}{2}}{1-j\Omega\frac{T_d}{2}}$$

rearranging we get

$$j\Omega\frac{T_d}{2} = \frac{e^{j\omega}-1}{e^{j\omega}+1} = \frac{e^{j\omega/2}(e^{+j\omega/2}-e^{-j\omega/2})}{e^{j\omega/2}(e^{+j\omega/2}+e^{-j\omega/2})}$$

$$= j\frac{\sin\ \omega/2}{\cos\ \omega/2}$$

or

$$\Omega = \frac{2}{T_d}\tan\ \omega/2 \tag{9.5}$$

or

$$\omega = 2\tan^{-1}\frac{\Omega T_d}{2} \tag{9.6}$$

The compression of frequency axis represented by (9.5) is nonlinear. This is illustrated in figure 9.4.

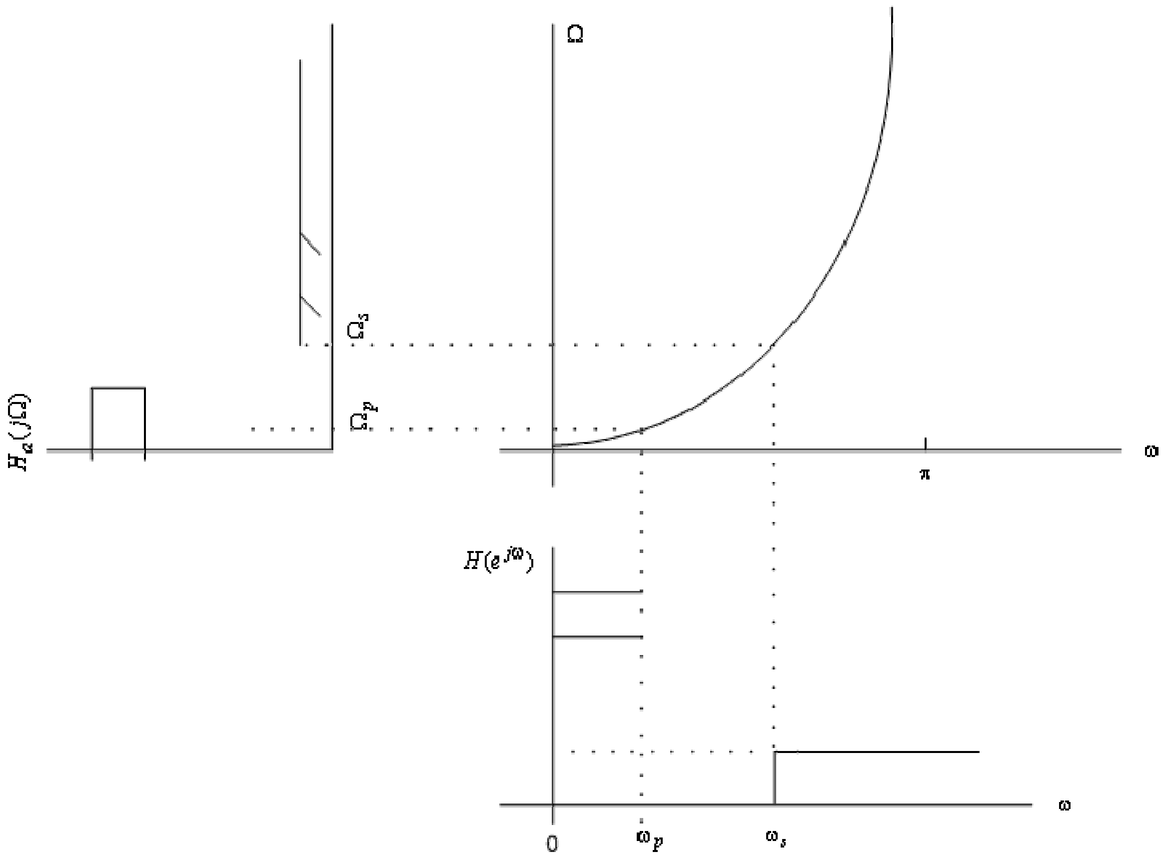


Fig 9.4

Because of the nonlinear compression of the frequency axis, there is considerable phase distortion in the bilinear transformation.

Example

We use the specifications given in the previous example. Using equation (9.5) with $T_d = 2$ we get

$$\Omega_p = \tan\frac{.25\pi}{2} = 0.414$$

$$\Omega = \tan\frac{.55\pi}{2}$$

Some frequently used analog filters

In the previous two examples we have used Butterworth filter. The Butterworth filter of order n is described by the magnitude square frequency response of

$$|H_n(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}$$

It has the following properties

1. $|H_n(j\Omega)|^2 = 1 at \Omega = 0$

2. $|H_n(j\Omega)|^2 = 1/2 \; at \; \Omega = \Omega_c$

3. $|H_n(j\Omega)|^2$ is monotonically decreasing function of Ω

4. As *n* gets larger, $|H_n(j\Omega)|^2$ approaches an ideal low pass filter

5. $|H_n(j\Omega)|^2$ is called maximally flat at origin, since all order derivative exist and they are zero at $\Omega = 0$

The poles of a Butterworth filter lie on circle of radius Ω_c in s-plane. There are two types of Chebyshev filters, one containing ripples in the passband (type I) and the other containing a ripple in the stopband (type II). A Type I low pass normalizer Chebyshev filter has the magnitude squared frequency response.

$$|H_n(j\Omega)|^2 = \frac{1}{1+\epsilon^2 \, T_n^2(\Omega)}$$

where $T_n\Omega$ is n^th order Chebyshev polynomial. We have the relationship

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n > 2$$

with $T_0(x) = 1, T_1(x) = x$.

Chebyshev filters have the following properties

1. The magnitude squared frequency response oscillates between 1 and $1/(1+\epsilon^2)$ within the passband, the so called equiripple and has a value of $1/(1+\epsilon^2)$ at $\Omega = 1$, the normalized cut off frequency.
2. The magnitude response is monotonic outside the passband including transitionand stopband.
3. The poles of the Chebysher filter lie on an ellipse in s-plane.

An elliptic filter has ripples both in passband and in stopband. The square magnitude frequency response is given by

$$|H_n(j\Omega)|^2 = \frac{1}{1+\epsilon^2 \, R_n^2(\Omega)}$$

where $R_n(\Omega)$ is Chebyshev rational function of O determined from specified ripple characteristics.

An n^{th} order Chebyshev filter has sharper cutoff than a Butterworth filter, that is, has a narrower transition bandwidth. Elliptic filter provides the smallest transition width.

Design of Digital filter using Digital to Digital transformation

There exists a set of transformation that takes a low pass digital filter and turn into highpass, bandpass, bandstop or another lowpass digital filter. These transformations are given in table 9.1.

The transformations all take the form of replacing the z^{-1} in $H(z)$ by $g(z^{-1})$ some function of.

Type From	To	Transformation	Design Formula

Low pass cutoff θ_p	Low pass cutoff ω_p	$z^{-1} \rightarrow \dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = \dfrac{\sin[(\theta_p - \omega_p)/2]}{\sin[(\theta_p + \omega_p)/2]}$
LPF	HPF	$z^{-1} \rightarrow \dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = -\dfrac{\cos[(\theta_p - \omega_p)/2]}{\cos[(\theta_p + \omega_p)/2]}$
LPF	BPF	$z^{-1} \rightarrow \dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos[(\omega_2 + \omega_1)/2]}{\cos[(\omega_2 - \omega_1)/2]}$ $k = \cot[(\omega_2 - \omega_1)/2] \tan \theta_p/2$
LPF	BSF	$z^{-1} \rightarrow \dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} \frac{1-k}{1+k}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos[(\omega_2 + \omega_1)/2]}{\cos[(\omega_2 - \omega_1)/2]}$ $k = \tan[(\omega_2 - \omega_1)/2] \tan \theta_p/2$

Starting with a set of digital specifications and using the inverse of the design equation given in table 9.1, a set of lowpass digital requirements can be established. A LPF digital prototype filter $H_p(z)$ is then selected to satisfy these requirements and the proper digital to digital transformation is applied to give the desired.

Example

Using the digital to digital transformation, find the system function $H(z)$ for a low-pass digital filter that satisfies the following set the requirements (a) monotone stop and passband (b)-3dB cutoff frequency of 0.5π (c) attenuation at and past 0.75π is at least 15dB.

Because of monotone requirement, a Butterworth filter is selected. The required n is given by

$$n = \frac{\log_{10}[(10^{-3}-1)/(10^{1.5}-1)]}{2\log_{10}\{[\tan(.5\pi/2)]/\tan .75\pi/2)\}} = 1.9412$$

rounded to 2.

$$\omega_p = 2\tan^{-1}[(10^{0.3})-1]^{-1/2n}\tan(.5\pi/2) = 0.5\pi$$

For $\theta_p = 1, u_p = .5$ we get from table 9.1. $\alpha = -.293$, From standard tables (or MATLAB) we find standard 2 *nd* order Butterworth filter with cut off $\theta_p = 1$ and then apply the digital transform to get

$$H(z) = \frac{(1+z^{-1})^2}{3.4142 + .5858z^{-2}}$$

FIR filter design

In the previous section, digital filters were designed to give a desired frequency response magnitude without regard to the phase response. In many cases a linear phase characteristics is required through the passband of the filter. It can be shown that causal IIR filter cannot produce a linear phase characteristics and only special forms of causal FIR filters can give linear phase.

If $\{h[n]\}$ represents the impulse response of a discrete time linear system a necessary and sufficient condition for linear phase is that $\{h[n]\}$ have finite duration *N* , that it be symmetric about its mid point, i.e.

$$h[n] = h[N-1-n], \quad n = 0,1,2, \dots (N-1)$$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{n}{2}-1} h[n]^{-j\omega n} + \sum_{n=N/2}^{N-1} h[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{N/2-1} h[n]^{-j\omega n} + \sum_{m}^{N/2-1} h[m]e^{-j\omega}(N-1-m)$$

For *N* even, we get

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2} \sum_{n=0}^{N/2-1} 2h[N]\cos(\omega(n-(N-1)/2))$$

For N odd

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2} \left\{ h[\frac{N-1}{2}] + \sum_{n=0}^{\frac{N-3}{2}} 2h[n]\cos[\omega(n-\frac{N-1}{2})] \right\}$$

For *N* even we get a non-integer delay, which will cause the value of the sequenceto change, [See continuous time implementation of discrete time system, for interpretation of non-integer delay].

One approach to design FIR filters with linear phase is to use windowing.

The easiest way to obtain an FIR filter is to simply truncate the impulse response of an IIR filter. If $\{h_d[n]\}$ is the impulse response of the designed FIR filter, then an FIR filter with impulseresponse $\{h[n]\}$ can be obtained as follows.

$$\{h[n]\} = \begin{cases} h_d[n], & N_1 \le n \le N_2 \\ 0, & otherwise \end{cases}$$

This can be thought of as being formed by a product of $\{h_d[n]\}$ and a window function $\{\omega[n]\}$

$$\{h[n]\} = \{h_d[n]\}\{\omega[n]\}$$

where $\{\omega[n]\}$ is said to be rectangular window and is given by

$$\{\omega[n]\} = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

Using modulation property of Fourier transfer

$$H(e^{j\omega}) = \frac{1}{2\pi}[H_d(e^{j\omega}) \oplus - W(e^{j\omega})]$$

For example if $H_d(e^{j\omega})$ is ideal low pass filter and $\{\omega[n]\}$ is rectangular window is measured version of the ideal low pass frequency response.

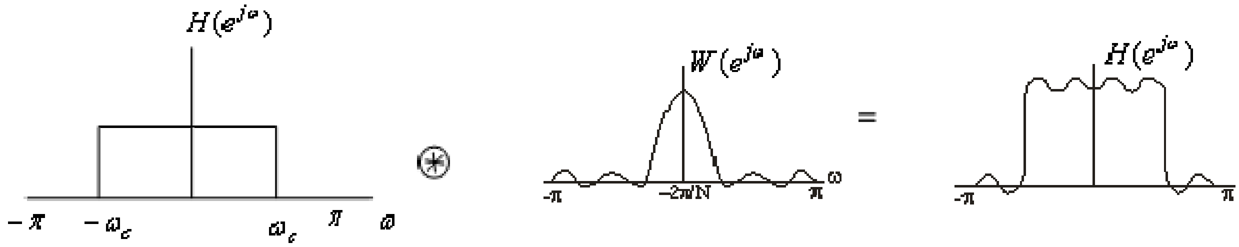


Fig 9.5

In general, the index the main lobe of $W(e^{j\omega})$, the more $H(e^{j\omega})$ spreading where as the narrower the main lobe (larger N), the closer $|H(e^{j\omega})|$ comes to. In general, we are left with a trade-off of making N large-enough so that smearing is minimized, yet small enough to allow reasonable implementation. Much work has been done on adjusting $\{\omega[n]\}$ to satisfy certain main lobe and side lobe requirements. Some of the commonly used windows are give in below.

(a) Rectangular

$$W_R(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

(b) Bartlett (or triangle)

$$\omega_B[n] = \begin{cases} 2n/(N-1), & 0 \le n \le (N-1)/2 \\ 2 - 2n/(N-1), & (N-1)/2 \le n \le N-1, \\ 0, & otherwise \end{cases}$$

(c) Hanning

$$\omega_{Ham}[n] = \begin{cases} \dfrac{1-\cos[2\pi n/(N-1)]}{2} & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

(d) Harming

$$\omega_{Ham}[n] = \begin{cases} 0.54 - 0.46\cos[2\pi n/(N-1)] & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

(e) Blackman

$$\omega_{Bl}[n] = \begin{cases} 0.42 - 0.5\cos[2\pi n/(N-1)] + 0.08\cos[4\pi n/(N-1)] & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

(f) Kaiser

$$\omega_k[n] = \begin{cases} \dfrac{I_0\left\{\omega_a\left[\left(\dfrac{N-1}{n}\right)^2 - \left(n - \dfrac{N-1}{n}\right)^2\right]^{1/2}\right\}}{I_0\left\{\omega_a\left(\dfrac{N-1}{2}\right)\right\}} & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

where $I_0(x)$ is modified zero-order Bessel function of the first kind given by

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\,d\theta$$

$$= 1 + \sum_{n=1}^{\infty}\left[\left(\frac{x}{2}\right)^n \frac{1}{n!}\right]^2$$

The main lobe width and first side lobe attenuation increase as we proceed down the window listed above.

An ideal lowpass filter with linear phase and cut off ω_c is characterized by

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The corresponding impulse response is

$$h_d[n] = \frac{\sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}$$

Since this is symmetric about $n = \alpha$, if we change $\alpha = (N-1)/2$ and use one of the windows listed above the will get linear phase FIR filter. Transition width and minimum stopped attenuation are listed in the Table 9.3.

Window	Transition Width	Minimum stopband attenuation
Rectangular	$4\pi/N$	-21db
Bartlett	$8\pi/N$	-25dB
Hanning	$8\pi/N$	-44dB
Hamming	$8\pi/N$	-53dB
Blackman	$12\pi/N$	-74dB
Kaiser	variable	variable

Table 9.3

We first choose a window that satisfies the minimum attenuation. The transition bandwidth is approximately that allows us to choose the value of N. Actual frequency response characteristic are then calculated and we see if the requirements are met or not. Accordingly N is adjusted parameters for kaiser window are obtained from design formula available for this MATLAB or similar programmes have all there formulas.

Realizations of Digital Filters

We have many realizations of digital filter. Some of these are now discussed. **Direct Form Realization** - An important class of linear time -invariant systems is characterized by the transfer function.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}}$$

A system with input $\{x[n]\}$ and output $\{y[n]\}$ could be realized by the following constant coefficient difference equation

$$\{y[n]\} = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{N} b_k x[n-k]$$

A realization of the filter using equation (9.31) is shown in figure (9.6)

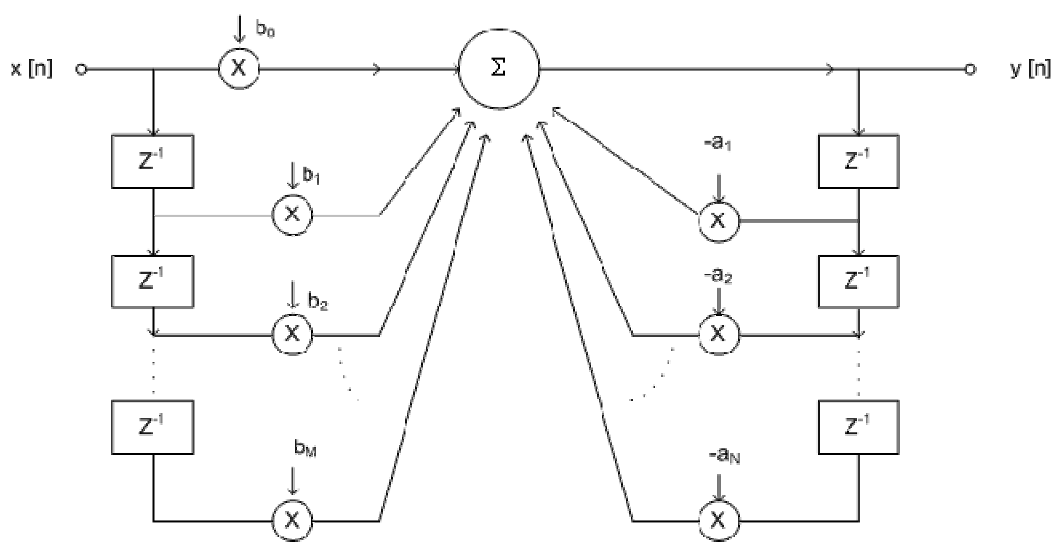


Fig 9.6 Direct form I

The output $y[n]$ is seen to be weighted sum of input $x[n]$ and past inputs $x[n-1], \ldots x[n-A]$ and past outputs. Another realization can be obtained by uniting $H(z)$ as product of two transfer functions $H_1(z)$ and $H_2(z)$, where $H_1(z)$ contains only the denominator or poles and $H_2(z)$ contains only the numerator or zeros as follows

$$H(z) = H_1(z)H_2(z)$$

where

$$H_1(z) = \frac{1}{1+\sum_{i=1}^{N} a_k z^{-k}}$$

$$H_2(z) = \sum_{k=0}^{N} b_k z^{-k}$$

Fig 9.7

The output of the filter is obtained by calculating the intermediate result $\{\omega[n]\}$ obtained from operating on the input with filter $H_1(z)$ and then operating on *w*[*n*] with filter.Thus we obtain

$$W(z) = X(z)H_1(z)$$

or

$$\omega[n] = x[n] - \sum_{k-1}^{N} a_k \omega[n-k]$$

and

$$Y(z) = W(z)H_2(z)$$

or

$$y[n] = \sum_{k=0}^{M} b_k \omega[n-k]$$

The realization is shown in figure 9.8

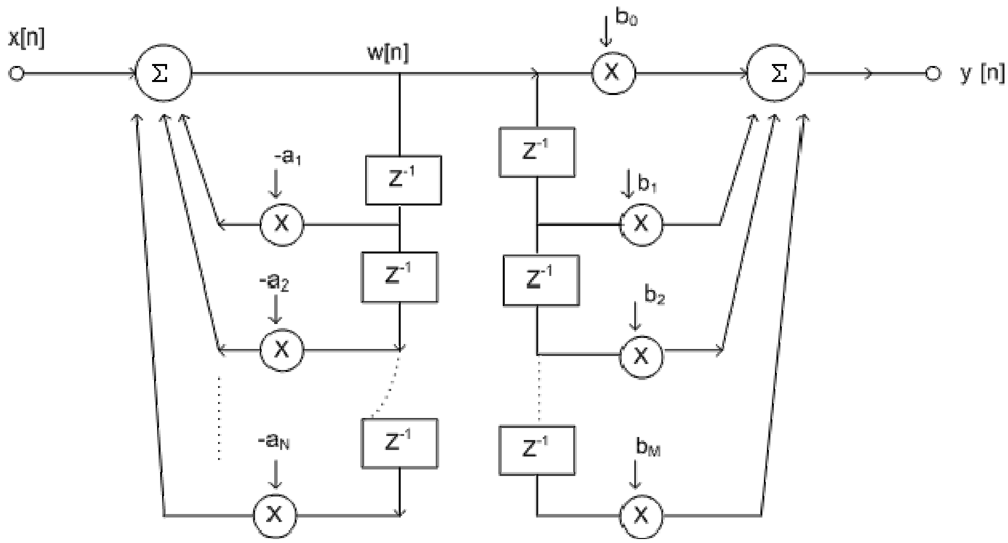


Fig 9.8

Upon close examination of Fig 9.8, it can be seen that the two branches of delay elements can be combined as they both refer to delayed versions of $\omega[n]$ and upon simplification, the direct form II canonical realization is obtained as shown in figure 9.9.

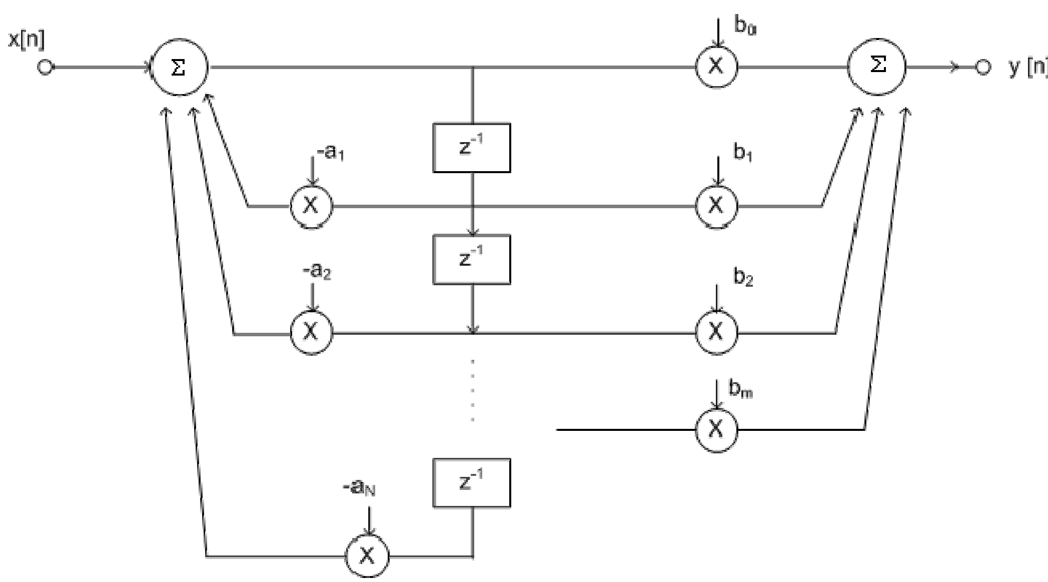


Fig 9.9 Direct form II

In this form the number of delay element is max (M,N). It can be shown that this is the minimum number of delay elements that are required to implement the digital filter. This does not mean that this is the best realization. Immunity to roundoff and quantization are very important considerations.

An important special case that is used as building block occurs when. Thus $H(z)$ is ratio of two qualities in z^{-1}, called biquadratic section, and is given by

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$= \frac{b_0 (1 + b_1^1 z^{-1} + b_2^1 z^{-2})}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

The alternative form is found to be useful for amplitude scaling for improving performance file filter operation. This form is shown in figure 9.10.

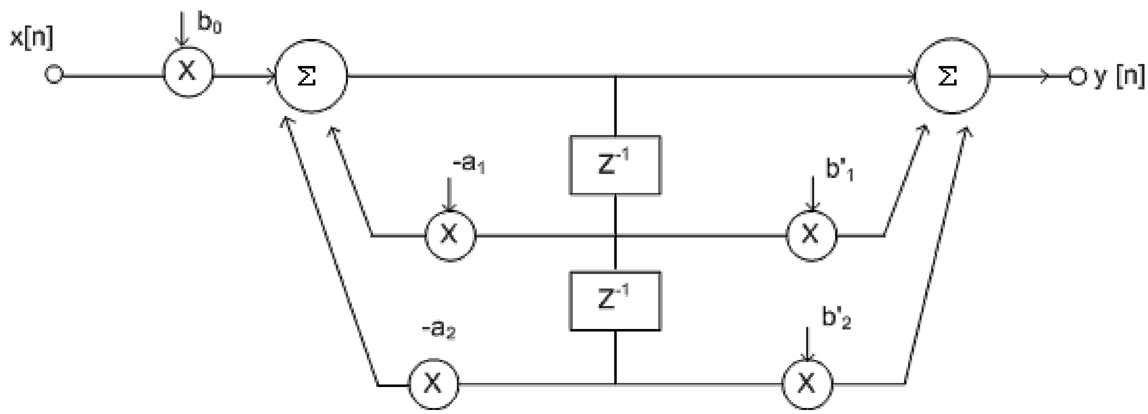


Fig 9.10

Cascade Realizations: In the cascade realization $H(z)$ is broken into productof transfer functions $H_1(z), H_2(z) \ldots H_k(z)$ each a rational expression in z^{-1} as follows

$$H(z) = H_1(z) H_2(z) \ldots H_k(z)$$

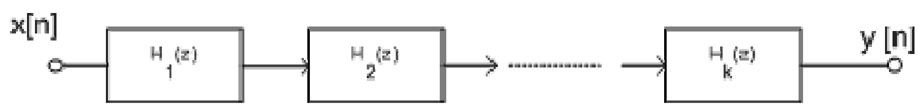


Fig 9.11

$H(z)$ could be broken up in many ways; however the most common method is to use biquadratic sections. Thus

$$H_k(z) = \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_1 k z^{-1} + a_2 k z^{-2}}, \; k = 1, 2, \ldots K$$

by letting b_{2k} and a_{2k} equal to zero we get bilinear section. Even among the biquadratic sections we have many choices as how we pair poles and zeros. Also the order of the sections can be different

Example:

Final cascade realization of

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{(z - .25)(z^2 - z + 0.5)}$$

Using only real coefficients $H(z)$ can be decompressed as

$$H(z) = \frac{8(z-.1899)(z^2-.31z+1.316)}{(z-.25)(z^2-z+0.5)}$$

Divides both numerator and denominator by z^3 and factoring 8 as 2×4, one possible rearrangement for $H(z)$ is

$$H(z) = \frac{(2-0.3799z^{-1})}{(1-0.25z-1)} \cdot \frac{(4-1.24z^{-1}+5.264z^{-2})}{(1-z^{-1}+.5z^{-2})}$$

This can be realized as shown is figure 9.12

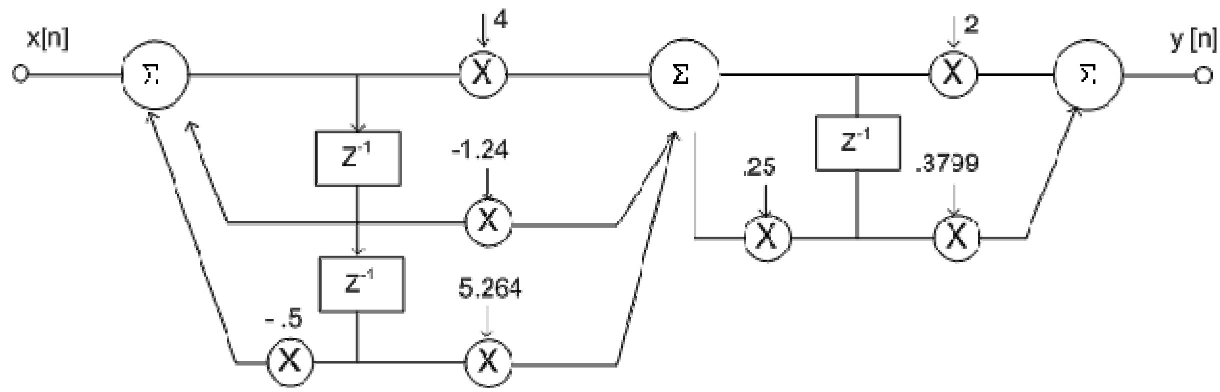


Fig 9.12

Parallel Realizations:

The transfer function *H* (*z*) could be written as a sum of transfer functions $H_1(z), H_2(z), - H_k(z)$ as follows:

$$H(z) = H_1(z) + H_2(z) + \ldots + H_k(z)$$

One parallel form results when $H_k(z)$ are all selected to be of the following form for $(M < N)$

$$H_k(z) = \frac{b_{0k} + b_{1k}z^{-1}}{1 + a_{k1}z - 1 + a_{k2}z - 2}, \; k = 1, 2, \ldots K$$

If $M \geq N$, we will have a section $H_0(z)$ of FIR filter, obtained by performing long division. Once denominator polynomial has degree more than the numerator polynomial we perform the partial fraction expansion. The resulting structure is shown in figure 9.13.

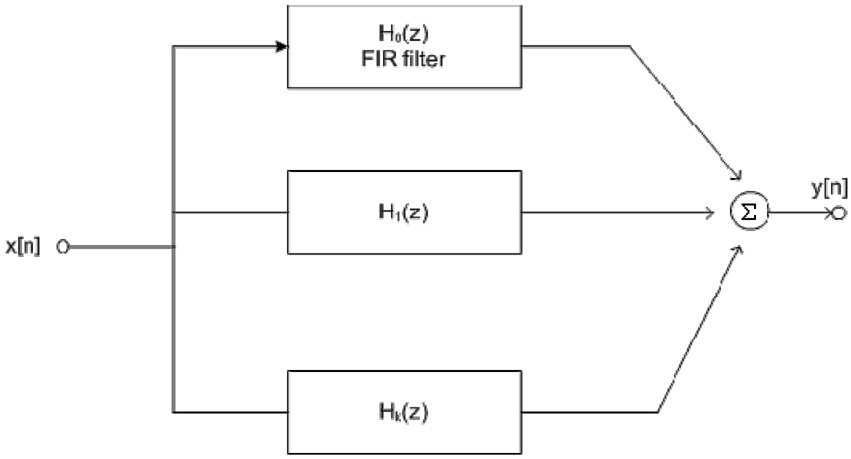


Fig 9.13

Example: Find the parallel form for the filter given in last example.

$$H(z) = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{(1 - .25z^{-1})(1 - z^{-1} + .5z^{-2})}$$

Using MATLAB program or otherwise we get

$$H(z) = 16 + \frac{8}{1 - .25z^{-1}} + \frac{-16 + 20z^{-1}}{1 - z^{-1} + .5z^{-2}}$$

using direst form realization for individual section we get the structure shown in figure 9.14.

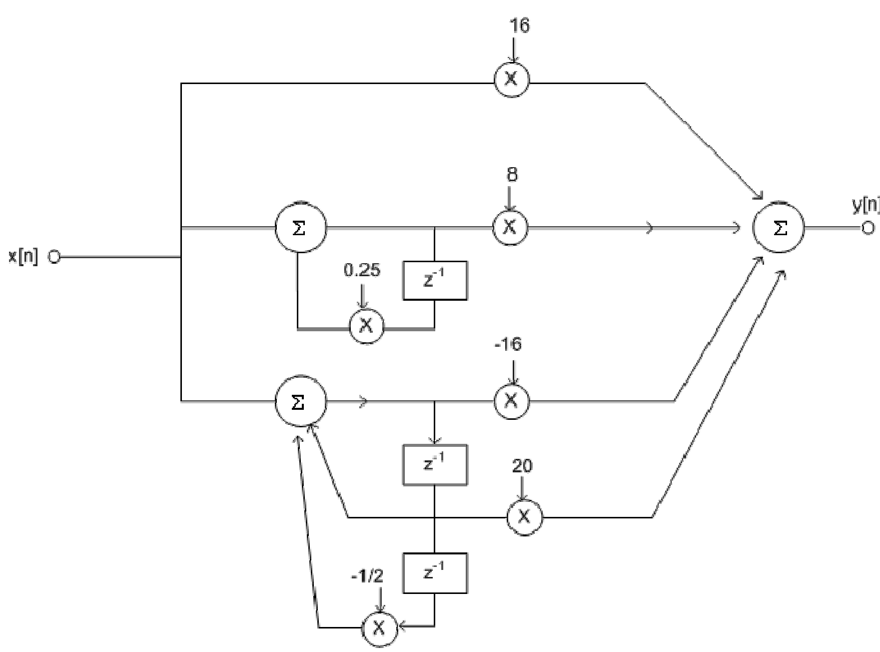


Fig 9.14 Apart from these there exist a number of other realizations like lattice form, state variable realization etc.